PERFECT DYADIC OPERATORS: WEIGHTED $T(1)$ THEOREM AND TWO WEIGHT ESTIMATES.

OLEKSANDRA BEZNOSOVA

Abstract. Perfect dyadic operators were first introduced in [1], where a local $T(b)$ theorem was proved for such operators. In [2] it was shown that for every singular integral operator T with locally bounded kernel on $\mathbb{R}^n \times \mathbb{R}^n$ there exists a perfect dyadic operator $\mathbb{T}$ such that $T - \mathbb{T}$ is bounded on $L^p(dx)$ for all $1 < p < \infty$.

In this paper we show a decomposition of perfect dyadic operators on real line into four well known operators: two selfadjoint operators, paraproduct and its adjoint. Based on this decomposition we prove a sharp weighted version of the $T(1)$ theorem for such operators, which implies A_2 conjecture for such operators with constant which only depends on $\|T(1)\|_{BMO^d}$, $\|T^*(1)\|_{BMO^d}$ and the constant in testing conditions for T. Moreover, the constant depends on these parameters at most linearly. In this paper we also obtain sufficient conditions for the two weight boundedness for a perfect dyadic operator and simplify these conditions under additional assumptions that weights are in the Muckenhoupt class A^d_∞.

1. Introduction

A perfect dyadic operator T is defined by:

$$Tf \;:=\; \int K(x,y)\, f(y)\, dy \;\text{ for }\; x \notin supp\, f,$$

where kernel K satisfies the following conditions:
standard size condition:

$$|K(x,y)| \;\leq\; \frac{1}{|x-y|} \tag{1.1}$$

and perfect cancellation condition:

$$\big|K(x,y) - K(x,y')\big| + \big|K(x,y) - K(x',y)\big| \;=\; 0 \tag{1.2}$$

whenever $x, x' \in I \in D$, $y, y' \in J \in D$, $I \cap J = \varnothing$. Where D is a set of dyadic intervals on the real line $D := \{I = [2^j k, 2^j(k+1)) : \; k, j \in \mathbb{Z}\}$.

Perfect dyadic operators appeared in the context of the local $T(b)$ theorems (see [1]) and later were extended to spaces of homogeneous type (see [3], [25]). Perfect dyadic operators are essentially Calderón–Zygmund singular integrals with singularity adapted to the fixed dyadic grid. Their main property is that any function supported on a dyadic cube with zero mean is mapped into the function supported on the same cube. In [2] it was shown that in $L^2(dx)$ perfect dyadic operators are a good approximation of Calderón–Zygmund singular integral operators.

2010 *Mathematics Subject Classification.* Primary 42B20, 42B25 ; Secondary 47B38.

Key words and phrases. Weighted norm estimate, Dyadic operators, Joint A_2-weights, Perfect dyadic operators, T(1) theorem.

The author was supported by the University of Alabama RGC grant.

In this paper we stick to the real line as a model case. In Section 2 we derive a very useful decomposition of the perfect dyadic operator into four well known operators: two selfadjoint operators, dyadic paraproduct and its adjoint. In particular, we show that a perfect dyadic operator T and its adjoint formally can be written as

$$\begin{aligned} Tf(x) &= \frac{1}{4}\sum_{I\in D}\left(K_I^+ + K_I^-\right)|I|\, m_I f\,\chi_I - \frac{1}{4}\sum_{I\in D}\left(K_I^+ - K_I^-\right)\sqrt{|I|}\,\langle f;h_I\rangle\,\chi_I \\ &\quad +\frac{1}{4}\sum_{I\in D}\left(K_I^+ - K_I^-\right)|I|^{3/2}\, m_I f\, h_I - \frac{1}{4}\sum_{I\in D}\left(K_I^+ + K_I^-\right)|I|\,\langle f;h_I\rangle\, h_I, \end{aligned} \tag{1.3}$$

$$\begin{aligned} T^*f(x) &= \frac{1}{4}\sum_{I\in D}\left(K_I^+ + K_I^-\right)|I|\, m_I f\,\chi_I(x) + \frac{1}{4}\sum_{I\in D}\left(K_I^+ - K_I^-\right)\sqrt{|I|}\,\langle f;h_I\rangle\,\chi_I(x) \\ &\quad -\frac{1}{4}\sum_{I\in D}\left(K_I^+ - K_I^-\right)|I|^{3/2} m_I f\, h_I(x) - \frac{1}{4}\sum_{I\in D}\left(K_I^+ + K_I^-\right)|I|\,\langle f;h_I\rangle\, h_I(x). \end{aligned} \tag{1.4}$$

For a given dyadic interval I, I^+ and I^- are its left and right halves. Coefficients K_I^+ and K_I^- are defined to be values of the kernel K of the perfect dyadic operator T on the dyadic cubes $I^+\times I^-$ and $I^-\times I^+$ respectively. The notation $m_I f$ stands for the average of the function f over the interval I, $m_I f := \frac{1}{|I|}\int_I f$.

In Section 3 using our decomposition we prove the $T(1)$ theorem and show several useful estimates on the kernel. We prove the following theorem.

Theorem 1. *Let T be a perfect dyadic operator that satisfies:*

(i) BMO conditions: $\|T(1)\|_{BMO^d}$ and $\|T^*(1)\|_{BMO^d}$ are bounded by a finite numerical constant Q;

(ii) Testing conditions: $\langle Th_I;h_I\rangle$ are uniformly bounded by Q for every dyadic interval $I\in D$.

Then T is bounded on L^2.

Moreover, T accepts decomposition (1.3) with coefficients $K_I^\pm$ satisfying size conditions:

$$\forall J\in D \qquad \left|K_J^+ + K_J^-\right||I| \;\le\; 2Q, \tag{1.5}$$

$T(1)$ conditions:

$$\forall J\in D \qquad \frac{1}{|J|}\sum_{I\in D(J)}\left(K_I^+ - K_I^-\right)^2|I|^3 \;\le\; 16Q \tag{1.6}$$

and testing conditions:

$$\forall J\in D \qquad \left|\frac{1}{|J|}\sum_{I\in D(J)}\left(K_I^+ + K_I^-\right)|I|^2\right| \;\le\; 8Q. \tag{1.7}$$

In Section 4 we 'lift' the above $T(1)$ theorem to the one weight case, $L^2(w)$, and provide an elementary proof of the A_2 conjecture for such operators; in particular we trace how the weighted norm of perfect dyadic operator depends on the dyadic BMO^d and testing constants of the perfect dyadic operator. We prove the following theorem.

Theorem 2. *Let T be a perfect dyadic operator on $\mathbb{R}$ such that*

$$\|T(1)\|_{BMO^d} \leqslant Q \quad \text{and} \quad \|T^*(1)\|_{BMO^d} \leqslant Q$$

and

$$\forall I \in D \quad \langle Th_I, h_I \rangle \leqslant Q$$

. Then T is bounded on $L^2(w)$ and

$$\|Tf\|_{L^2(w)} \leqslant CQ[w]_{A_2^d}\|f\|_{L^2(w)}, \tag{1.8}$$

with some constant C independent of the operator T.

In particular, by [14], this implies that under assumptions of the theorem a perfect dyadic operator T is also bounded in $L^p(w)$ for all $w \in A_p^d$, its norms are bounded by $C[w]_{A_p^d}^{\max\{1,\frac{1}{p}\}}$, and dependence $C(Q)$ can be traced as well.

To the best of our knowledge, such sharp weighted version of $T(1)$ theorem is new. In particular it is interesting that the constant in (1.8) depends only on the BMO^d norms of $T(1)$ and $T^*(1)$ and on the constant in testing conditions. Moreover, the dependence is at most linear.

In Section 5 we go even further and give sufficient conditions for the two weight boundedness for a single perfect dyadic operator. Currently there are two schools of thought regarding the two weight problem. First, given one operator T find necessary and sufficient conditions on the weights to ensure boundedness of the operator on the appropriate spaces. Second, given a family of operators find necessary and sufficient conditions on the weights to ensure boundedness of the family of operators. In both cases we are mostly interested in Calderón–Zygmund singular integral operators. In the first case, the conditions are usually "testing conditions" obtained from checking boundedness of the given operator on a collection of test functions. In the second case, the conditions are more "geometric", meaning they seem to only involve the weights and not the operators, such as Carleson conditions or bilinear embedding conditions, Muckenhoupt A_2 type conditions or bumped conditions. Operators of interest are the maximal function [31, 26, 30, 35], fractional and Poisson integrals [32, 10], the Hilbert transform [8, 9, 19, 28, 23, 20] and general Calderón–Zygmund singular integral operators and their commutators [13, 12, 11, 17, 27], the square functions [21, 22], paraproducts and their dyadic counterparts. Necessary and sufficient conditions are only known for the maximal function, fractional and Poisson integrals [31], square functions [21] and the Hilbert transform [20, 23], and among the dyadic operators for the martingale transform, the dyadic square functions, positive and well localized dyadic operators [37, 28, 29, 34, 15, 16, 18, 24, 33, 36].

We prove the following sufficient conditions for the two weight boundedness of the perfect dyadic operators.

Theorem 3. *Let (v, u) be a pair of measurable functions, such that u and v^{-1}, the reciprocal of v, are weights on $\mathbb{R}$ such that*

$$(v, u) \in A_2^d\,, \text{i.e. } [v, u]_{A_2^d} := \sup_{I \in D} m_I(v^{-1}) m_I u < \infty, \tag{1.9}$$

$$\forall J \in D \quad \frac{1}{|J|} \sum_{I \in D(J)} (\Delta_I u)^2 m_I(v^{-1})|I| \leqslant C m_I u, \tag{1.10}$$

$$\forall J \in D \quad \frac{1}{|J|} \sum_{I \in D(J)} (\Delta_I(v^{-1}))^2 m_I u |I| \leqslant C m_I(v^{-1}), \tag{1.11}$$

and operator T_0 is bounded from $L^2(v)$ in $L^2(u)$, i.e.

$$\forall J \in D \quad \frac{1}{|J|} \int_J \left(\sum_{I \in D(J)} \left| \frac{\Delta_I u \Delta_I(v^{-1})}{m_I u} \right| \chi_I(x) \right)^2 u(x)dx \leqslant C m_J(v^{-1}) \tag{1.12}$$

and

$$\forall J \in D \quad \frac{1}{|J|} \int_J \left(\sum_{I \in D(J)} \left| \frac{\Delta_I u \Delta_I(v^{-1})}{m_I(v^{-1})} \right| \chi_I(x) \right)^2 v^{-1}(x)dx \leqslant C m_J u. \tag{1.13}$$

Let T be a perfect dyadic operator with $\|T(1)\|_{BMO^d}, \|T^*(1)\|_{BMO^d} \leqslant Q$ and for every $I \in D$ we have $\langle T h_I; h_I \rangle \leqslant Q$.

Then T is bounded from $L^2(v)$ to $L^2(u)$ whenever for any dyadic interval $J \in D$

$$\frac{1}{|J|} \sum_{I \in D(J)} (K_I^+ + K_I^-)|I|^2 m_I u \leqslant C m_J u, \tag{1.14}$$

$$\frac{1}{|J|} \sum_{I \in D(J)} (K_I^+ + K_I^-)|I|^2 m_I(v^{-1}) \leqslant C m_J(v^{-1}), \tag{1.15}$$

$$\frac{1}{|J|} \sum_{I \in D(J)} \frac{(K_I^+ - K_I^-)^2 |I|^3}{m_I u} \leq C m_J(v^{-1}) \tag{1.16}$$

and

$$\frac{1}{|J|} \sum_{I \in D(J)} \frac{(K_I^+ - K_I^-)^2 |I|^3}{m_I(v^{-1})} \leq C m_J u. \tag{1.17}$$

When both weights are in the Muckenhoupt class A_∞^d we simplify conditions of Theorem 3 to the following.

Theorem 4. *Let (v, u) be a pair of measurable functions, such that u and v^{-1}, the reciprocal of v, are A_∞^d weights on $\mathbb{R}$ such that*

$$(v, u) \in A_2^d, \text{ i.e. } [v, u]_{A_2^d} := \sup_{I \in D} m_I(v^{-1}) m_I u < \infty, \tag{1.18}$$

and operator T_0 is bounded from $L^2(v)$ in $L^2(u)$, i.e.

$$\forall J \in D \quad \frac{1}{|J|} \int_J \left(\sum_{I \in D(J)} \left| \frac{\Delta_I u \Delta_I(v^{-1})}{m_I u} \right| \chi_I(x) \right)^2 u(x)dx \leqslant C m_J(v^{-1}) \tag{1.19}$$

and

$$\forall J \in D \quad \frac{1}{|J|} \int_J \left(\sum_{I \in D(J)} \left| \frac{\Delta_I u \Delta_I(v^{-1})}{m_I(v^{-1})} \right| \chi_I(x) \right)^2 v^{-1}(x)dx \leqslant C m_J u. \tag{1.20}$$

Let T be a perfect dyadic operator with $\|T(1)\|_{BMO^d}, \|T^*(1)\|_{BMO^d} \leqslant Q$ and for every $I \in D$ we have $\langle Th_I; h_I\rangle \leqslant Q$.

Then T is bounded from $L^2(v)$ to $L^2(u)$ whenever for any dyadic interval $J \in D$

$$\frac{1}{|J|}\sum_{I\in D(J)} \frac{(K_I^+ - K_I^-)^2|I|^3}{m_I u} \le C m_J(v^{-1}) \tag{1.21}$$

and

$$\frac{1}{|J|}\sum_{I\in D(J)} \frac{(K_I^+ - K_I^-)^2|I|^3}{m_I(v^{-1})} \le C m_J u. \tag{1.22}$$

The main novelty of this paper is the decomposition, very strong quantitative estimates in the A_2 conjecture and sufficient conditions for the two weight boundedness of perfect dyadic operators.

2. Perfect dyadic operator on $\mathbb{R}$: definition and formal decomposition.

Let T be a perfect dyadic singular integral operator, i.e an operator defined by:

$$Tf := \int K(x,y)\, f(y)\, dy \;\text{ for }\; x \notin supp\, f, \tag{2.23}$$

where kernel K satisfies the following conditions:
standard size condition:

$$|K(x,y)| \le \frac{1}{|x-y|} \tag{2.24}$$

and perfect cancelation condition:

$$\big|K(x,y) - K(x,y')\big| + \big|K(x,y) - K(x',y)\big| = 0 \tag{2.25}$$

whenever $x, x' \in I \in D$, $y, y' \in J \in D$, $I \cap J = \varnothing$.

Note that, by perfect dyadic cancellation condition, $K(x,y)$ is constant on $I^+ \times I^-$ and $I^- \times I^+$ for any dyadic interval $I \in D$. We define

$$K_I^+ := K(x,y), \qquad x \in I^+,\; y \in I^-$$

and

$$K_I^- := K(x,y), \qquad x \in I^-,\; y \in I^+.$$

Then

$$K(x,y) = \sum_{I\in D} K_I^+ \chi_{I^+\times I^-} + K_I^- \chi_{I^-\times I^+}.$$

Then perfect dyadic singular operator can be written as

$$Tf(x) = \sum_{I\in D} K_I^+ f_{I^-}\chi_{I^+}(x) + K_I^- f_{I^+}\chi_{I^-}(x), \tag{2.26}$$

where $f_I := \int_I f(x)dx$.

Now we are going to rewrite T in a more convenient form. It is easy to see that

$$\begin{aligned} & K_I^+ f_{I^-}\chi_{I^+}(x) + K_I^- f_{I^+}\chi_{I^-}(x) \\ = {} & \frac{K_I^+ f_{I^-} + K_I^- f_{I^+}}{2}(\chi_{I^+}(x) + \chi_{I^-}(x)) + \frac{K_I^+ f_{I^-} - K_I^- f_{I^+}}{2}(\chi_{I^+}(x) - \chi_{I^-}(x)) \end{aligned}$$

$$(2.27)\quad = \frac{1}{2}\left(K_I^+ f_{I^-} + K_I^- f_{I^+}\right)\chi_I(x) + \frac{1}{2}\left(K_I^+ f_{I^-} - K_I^- f_{I^+}\right)\sqrt{|I|}\,h_I(x),$$

where $\{h_I\}_{I\in D}$ is the Haar system of functions, normalized in L^2, $h_I := |I|^{-1/2}\left(\chi_{I^+} - \chi_{I^-}\right)$. Thus, $Tf(x)$ can be written as:

$$Tf(x) = \sum_{I\in D} \frac{1}{2}\left(K_I^+ f_{I^-} + K_I^- f_{I^+}\right)\chi_I + \frac{1}{2}\left(K_I^+ f_{I^-} - K_I^- f_{I^+}\right)h_I\sqrt{|I|}. \tag{2.28}$$

Now let us handle the coefficients $\left(K_I^+ f_{I^-} + K_I^- f_{I^+}\right)$ and $\left(K_I^+ f_{I^-} - K_I^- f_{I^+}\right)$:

$$\begin{aligned} K_I^+ f_{I^-} + K_I^- f_{I^+} &= K_I^+ \int_{I^-} f + K_I^- \int_{I^+} f \\ &= \frac{1}{2}\left(K_I^+ + K_I^-\right)\left(\int_{I^-} f + \int_{I^+} f\right) - \frac{1}{2}\left(K_I^+ - K_I^-\right)\left(\int_{I^+} f - \int_{I^-} f\right) \\ &= \frac{1}{2}\left(K_I^+ + K_I^-\right)|I|\; m_I f - \frac{1}{2}\left(K_I^+ - K_I^-\right)\sqrt{|I|}\,\langle f; h_I\rangle \qquad (2.29) \end{aligned}$$

and, similarly, replacing K_I^- by $-K_I^-$,

$$K_I^+ f_{I^-} - K_I^- f_{I^+} = \frac{1}{2}\left(K_I^+ - K_I^-\right)|I|\; m_I f - \frac{1}{2}\left(K_I^+ + K_I^-\right)\sqrt{|I|}\,\langle f; h_I\rangle\,. \tag{2.30}$$

We plug (2.29) and (2.30) in (2.28) and obtain a formal representation of the perfect dyadic operator T:

$$\begin{aligned} Tf(x) &= \frac{1}{4}\sum_{I\in D}\left(K_I^+ + K_I^-\right)|I|\; m_I f\,\chi_I - \left(K_I^+ - K_I^-\right)\sqrt{|I|}\,\langle f; h_I\rangle\,\chi_I \\ &\qquad + \left(K_I^+ - K_I^-\right)|I|^{3/2}\; m_I f\, h_I - \left(K_I^+ + K_I^-\right)|I|\,\langle f; h_I\rangle\, h_I. \qquad (2.31) \end{aligned}$$

We can represent T^* in a similar way:

$$\begin{aligned} T^* f(x) &= \sum_{I\in D} K_I^- f_{I^-}\chi_{I^+}(x) + K_I^+ f_{I^+}\chi_{I^-}(x) \\ &= \sum_{I\in D} \frac{1}{2}\left(K_I^- f_{I^-} + K_I^+ f_{I^+}\right)\chi_I(x) + \frac{1}{2}\left(K_I^- f_{I^-} - K_I^+ f_{I^+}\right)\sqrt{|I|}\,h_I(x) \\ &= \frac{1}{4}\sum_{I\in D}\left(K_I^+ + K_I^-\right)|I|\; m_I f\,\chi_I(x) + \left(K_I^+ - K_I^-\right)\sqrt{|I|}\,\langle f; h_I\rangle\,\chi_I(x) \\ &\qquad - \left(K_I^+ - K_I^-\right)|I|^{3/2}\, m_I f\, h_I(x) - \left(K_I^+ + K_I^-\right)|I|\,\langle f; h_I\rangle\, h_I(x). \qquad (2.32) \end{aligned}$$

3. Decomposition of T, additional assumptions.

In this section we prove Theorem 1 modulo the unweighted boundedness of the operator T_1, which is done in a more general weighted case in the next section.

We assume, in addition to T being perfect dyadic, that $T(1)$ and $T^*(1)$ are both in the dyadic BMO^d, with the BMO^d norm of at most Q. We will also assume that T satisfies testing conditions $\langle Th_I, h_I\rangle \leqslant Q$ for all dyadic intervals I.

3.1. **The BMO^d condition.** A function $b(x)$, or a sequence $\{b_I\}_{I\in D} = \{\langle b; h_I\rangle\}_{I\in D}$ is in the dyadic BMO^d whenever the sequence b_I^2 is a Carleson sequence, i.e. there is a finite constant C such that for all dyadic intervals $J \in D$ we have

$$\frac{1}{|J|}\sum_{I\in D(J)} b_I^2 \leqslant C.$$

The best such constant C is called the dyadic BMO^d norm of b.

Let $T(1)$, $T^*(1) \in BMO^d$. We want to show that

$$\frac{1}{|J|}\sum_{I\in D(J)} (K_I^+ - K_I^-)^2|I|^3 \leqslant 16Q.$$

First let us plug $f(x) = 1(x)$ in T and T^*:

$$T(1) \;=\; \frac{1}{4}\sum_{I\in D}\left(K_I^+ + K_I^-\right)|I|\,\chi_I(x) + \left(K_I^+ - K_I^-\right)|I|^{3/2}\,h_I(x)$$

and

$$T^*(1) \;=\; \frac{1}{4}\sum_{I\in D}\left(K_I^+ + K_I^-\right)|I|\,\chi_I(x) - \left(K_I^+ - K_I^-\right)|I|^{3/2}\,h_I(x).$$

A function $f(x)$ belongs to the dyadic class BMO^d whenever $\sup_{J\in D}\frac{1}{|J|}\sum_{I\in D(J)}\langle f; h_I\rangle^2 \leq Q$, the smallest such constant is the BMO^d-norm of the function $f(x)$. So, let us find $\langle T(1); h_J\rangle$ and $\langle T^*(1); h_J\rangle$:

$$\begin{aligned}\langle T(1); h_J\rangle \;&=\; \frac{1}{4}\sum_{I\in D}\left(K_I^+ + K_I^-\right)|I|\,\langle\chi_I(x); h_J(x)\rangle + \left(K_I^+ - K_I^-\right)|I|^{3/2}\,\langle h_I(x); h_J(x)\rangle\\ &=\; \frac{1}{4}\sum_{I\in D(J^+)}\left(K_I^+ + K_I^-\right)|I|\,\frac{|I|}{\sqrt{|J|}} - \frac{1}{4}\sum_{I\in D(J^-)}\left(K_I^+ + K_I^-\right)|I|\,\frac{|I|}{\sqrt{|J|}}\\ &\quad +\frac{1}{4}\left(K_J^+ - K_J^-\right)|J|^{3/2}. \end{aligned}\tag{3.33}$$

Similarly,

$$\begin{aligned}\langle T^*(1); h_J\rangle \;&=\; \frac{1}{4}\sum_{I\in D}\left(K_I^+ + K_I^-\right)|I|\,\langle\chi_I(x); h_J(x)\rangle - \left(K_I^+ - K_I^-\right)|I|^{3/2}\,\langle h_I(x); h_J(x)\rangle\\ &=\; \frac{1}{4}\sum_{I\in D(J^+)}\left(K_I^+ + K_I^-\right)|I|\,\frac{|I|}{\sqrt{|J|}} - \frac{1}{4}\sum_{I\in D(J^-)}\left(K_I^+ + K_I^-\right)|I|\,\frac{|I|}{\sqrt{|J|}}\\ &\quad -\frac{1}{4}\left(K_J^+ - K_J^-\right)|J|^{3/2}. \end{aligned}\tag{3.34}$$

Let

$$\alpha_J \;:=\; \sum_{I\in D(J^+)}\left(K_I^+ + K_I^-\right)\frac{|I|^2}{\sqrt{|J|}} - \sum_{I\in D(J^-)}\left(K_I^+ + K_I^-\right)\frac{|I|^2}{\sqrt{|J|}}$$

and

$$\beta_J \;:=\; \left(K_J^+ - K_J^-\right)|J|^{3/2},$$

then

$$\langle T(1); h_J\rangle \;=\; \frac{1}{4}\left(\alpha_J+\beta_J\right), \qquad \langle T^*(1); h_J\rangle \;=\; \frac{1}{4}\left(\alpha_J-\beta_J\right).$$

If $T(1)\in BMO^d$ with $\|T(1)\|_{BMO^d}\le Q$, then for every dyadic interval $J\in D$

$$\frac{1}{|J|}\sum_{I\in D(J)}\langle T(1); h_I\rangle^2 \;=\; \frac{1}{16\,|J|}\sum_{I\in D(J)}\left(\alpha_I+\beta_I\right)^2 \;\le\; Q. \tag{3.35}$$

Similarly, $T^*(1)\in BMO^d$ with $\|T^*(1)\|_{BMO^d}\le Q$ implies that

$$\forall J\in D \quad \frac{1}{|J|}\sum_{I\in D(J)}\langle T^*(1); h_I\rangle^2 \;=\; \frac{1}{16\,|J|}\sum_{I\in D(J)}\left(\alpha_I-\beta_I\right)^2 \;\le\; Q. \tag{3.36}$$

Let us see how conditions (3.35) and (3.36) imply that

$$\forall J\in D \quad \frac{1}{|J|}\sum_{I\in D(J)}\beta_I^2 \;=\; \frac{1}{|J|}\sum_{I\in D}\left(K_I^+-K_I^-\right)^2|I|^3 \;\le\; 16Q.$$

The sum $\frac{1}{|J|}\sum_{I\in D(J)}\beta_I^2$ can be written as:

$$\begin{aligned}
\frac{1}{|J|}\sum_{I\in D(J)}\beta_I^2 \;&=\; \frac{1}{|J|}\sum_{I\in D(J)}\left(\frac{\beta_I}{2}-\frac{\alpha_I}{2}+\frac{\beta_I}{2}+\frac{\alpha_I}{2}\right)^2\\
&=\; \frac{1}{|J|}\sum_{I\in D(J)}\left(\frac{\beta_I}{2}-\frac{\alpha_I}{2}\right)^2+\frac{2}{|J|}\sum_{I\in D(J)}\left(\frac{\beta_I}{2}-\frac{\alpha_I}{2}\right)\left(\frac{\beta_I}{2}+\frac{\alpha_I}{2}\right)+\frac{1}{|J|}\sum_{I\in D(J)}\left(\frac{\beta_I}{2}+\frac{\alpha_I}{2}\right)^2\\
&\le\; 8Q+\frac{1}{2\,|J|}\sum_{I\in D(J)}\left(\beta_I-\alpha_I\right)\left(\beta_I+\alpha_I\right).
\end{aligned}$$

By Cauchy-Schwarz

$$\frac{1}{|J|}\sum_{I\in D(J)}\left(\beta_I-\alpha_I\right)\left(\beta_I+\alpha_I\right) \;\le\; \left(\frac{1}{|J|}\sum_{I\in D(J)}\left(\beta_I-\alpha_I\right)^2\right)^{\frac{1}{2}}\left(\frac{1}{|J|}\sum_{I\in D(J)}\left(\beta_I+\alpha_I\right)^2\right)^{\frac{1}{2}} \;\le\; 16Q.$$

Hence, for every dyadic interval $J\in D$

$$\frac{1}{|J|}\sum_{I\in D(J)}\beta_I^2 \;=\; \frac{1}{|J|}\sum_{I\in D}\left(K_I^+-K_I^-\right)^2|I|^3 \;\le\; 16Q. \tag{3.37}$$

3.2. **Testing conditions.** We assume that T satisfies testing conditions

$$\forall J\in D \qquad \langle Th_J; h_J\rangle \;\le\; Q. \tag{3.38}$$

Let us write $\langle Th_J; h_J\rangle$ as follows:

$$\begin{aligned}
\left|\langle Th_J; h_J\rangle\right| \;&=\; \frac{1}{4}\Bigg|\sum_{I\in D}\left(K_I^++K_I^-\right)\langle h_J;\chi_I\rangle\langle\chi_I;h_J\rangle-\left(K_I^+-K_I^-\right)\sqrt{|I|}\,\langle h_J;h_I\rangle\langle\chi_I;h_J\rangle\\
&\quad+\left(K_I^+-K_I^-\right)\sqrt{|I|}\,\langle h_J;\chi_I\rangle\langle h_I;h_J\rangle-\left(K_I^++K_I^-\right)|I|\,\langle h_J;h_I\rangle\langle h_I;h_J\rangle\Bigg|
\end{aligned}$$

$$
\begin{aligned}
&= \left| \frac{1}{4} \sum_{I \in D(J^+)} \left(K_I^+ + K_I^-\right) \left(\frac{|I|}{\sqrt{|J|}}\right)^2 + \frac{1}{4} \sum_{I \in D(J^-)} \left(K_I^+ + K_I^-\right) \left(-\frac{|I|}{\sqrt{|J|}}\right)^2 \right. \\
&\quad \left. -\frac{1}{4}\left(K_J^+ + K_J^-\right)|J| \right| \\
&= \left| \frac{1}{4|J|} \sum_{I \in D(J)} \left(K_I^+ + K_I^-\right)|I|^2 - \frac{1}{2}\left(K_J^+ + K_J^-\right)|J| \right| \ \le \ Q.
\end{aligned}
$$

Note that by standard size condition on the kernel K, we know that $\left|K_J^+ + K_J^-\right| |J| \le 2Q$, so

$$
\left| \frac{1}{|J|} \sum_{I \in D(J)} \left(K_I^+ + K_I^-\right)|I|^2 \right| \ \le \ 8Q. \tag{3.39}
$$

So, a perfect dyadic Calderón-Zygmund singular integral operator T satisfying the $T(1)$ conditions $\|T(1)\|_{BMO^2}, \|T^*(1)\|_{BMO^d} \leqslant Q$ and the dyadic testing conditions ($\langle Th_J; h_J\rangle \le Q \quad \forall J \in D$), can be written as:

$$
\begin{aligned}
Tf(x) &= \sum_{I \in D} K_I^+ f_{I^-} \chi_{I^+}(x) + K_I^- f_{I^+} \chi_{I^-}(x) \\
&= \frac{1}{4} \sum_{I \in D} \left(K_I^+ + K_I^-\right)|I|\, m_I f\, \chi_I(x) - \left(K_I^+ - K_I^-\right)\sqrt{|I|}\,\langle f; h_I\rangle\, \chi_I(x) \\
&\quad + \left(K_I^+ - K_I^-\right)|I|^{3/2}\, m_I f\, h_I(x) - \left(K_I^+ + K_I^-\right)|I|\,\langle f; h_I\rangle\, h_I(x).
\end{aligned} \tag{3.40}
$$

Moreover, coefficients K_I^+ and K_I^- satisfy the following conditions:
size conditions

$$
\forall J \in D \qquad \left|K_J^+ + K_J^-\right| |I| \ \le \ 2Q, \tag{3.41}
$$

$T(1)$ conditions

$$
\forall J \in D \qquad \frac{1}{|J|} \sum_{I \in D(J)} \left(K_I^+ - K_I^-\right)^2 |I|^3 \ \le \ 16Q \tag{3.42}
$$

and testing conditions

$$
\forall J \in D \qquad \left| \frac{1}{|J|} \sum_{I \in D(J)} \left(K_I^+ + K_I^-\right)|I|^2 \right| \ \le \ 8Q. \tag{3.43}
$$

By (3.40) operator T can be written as

$$
T = \frac{1}{4}\left(T_1 - T_2 + T_3 - T_4\right), \tag{3.44}
$$

where

$$
T_1 f(x) \ = \ \sum_{I \in D} \left(K_I^+ + K_I^-\right)|I|\, m_I f\, \chi_I(x), \tag{3.45}
$$

$$T_2 f(x) \;=\; \sum_{I\in D} \left(K_I^+ - K_I^-\right) \sqrt{|I|}\, \langle f; h_I\rangle\, \chi_I(x), \tag{3.46}$$

$$T_3 f(x) \;=\; \sum_{I\in D} \left(K_I^+ - K_I^-\right) |I|^{3/2}\, m_I f\, h_I(x), \tag{3.47}$$

$$T_4 f(x) \;=\; \sum_{I\in D} \left(K_I^+ + K_I^-\right) |I|\, \langle f; h_I\rangle\, h_I(x), \tag{3.48}$$

since all four operators are known well-defined linear operators bounded on L^2. Operators T_3 and T_2 are dyadic paraproduct and its adjoint, that are both well defined and bounded in L^2 since the sequence $b_I = (K_I^+ - K_I^-)|I|^{3/2}$ satisfies $\sum_{I\in D(J)} b_I^2 \leqslant C|J|$ for any $J \in D$ by (3.42). T_4 is a martingale transform with symbol bounded uniformly by (3.41). In the next section we show that T_1 is bounded on L^2 and on all weighted spaces for weights in A_2^d.

This completes the proof of Theorem 1.

4. Weighted $T(1)$ theorem for perfect dyadic operators

Let w be a weight, i.e. almost everywhere positive locally integrable function on the real line. Let a weight w be such that w^{-1} is a weight as well. Assume that w is in the dyadic Muckenhoupt class A_2^d, i.e.

$$[w]_{A_2^d} := \sup_{I\in D} m_I w\, m_I(w^{-1}) \; < \infty.$$

In order to show that the weighted $L^2(w)$ norm of the operator T depends on the A_2-constant of the weight w at most linearly, it is enough to show the linear bound for each of the operators T_i, $i = 1, 2, 3, 4$.

First let us show that T_1 is bounded on $L^2(w)$ and its norm depends on the A_2-constant of the weight w at most linearly. We will do it by duality, we will show that $\forall f \in L^2(w)$ and $\forall g \in L^2(w^{-1})$

$$\langle T_1 f; g\rangle \;=\; \sum_{I\in D} \left(K_I^+ + K_I^-\right) |I|^2\, m_I f\, m_I g \;\le\; C[w]_{A_2^d}\, \|f\|_{L^2(w)}\, \|g\|_{L^2(w^{-1})}$$

or, alternatively, $\forall f, g \in L^2$

$$\sum_{I\in D} \left(K_I^+ + K_I^-\right) |I|^2\, m_I\left(f w^{-1/2}\right)\, m_I\left(g w^{1/2}\right) \;\le\; C[w]_{A_2^d}\, \|f\|_{L^2}\, \|g\|_{L^2}\,. \tag{4.49}$$

Without loss of generality we may assume that coefficients $(K_I^+ + K_I^-)$ are all non-negative. We are going to use the following version of the bilinear embedding theorem from [28].

Theorem 5 (Nazarov, Treil, Volberg)**.** *Let v and w be weights. Let $\{a_I\}$ be a sequence of nonnegative numbers, s.t. for all dyadic intervals $J \in D$ the following three inequalities hold with some constant $Q > 0$:*

$$\frac{1}{|J|} \sum_{I\in D(J)} a_I\, m_I w\, m_I v \;\le\; Q, \tag{4.50}$$

$$\frac{1}{|J|} \sum_{I\in D(J)} a_I\, m_I w \;\le\; Q\, m_I w, \tag{4.51}$$

$$\frac{1}{|J|}\sum_{I\in D(J)} a_I\, m_I v \;\le\; Q\, m_I v, \tag{4.52}$$

then for any two nonnegative functions $f, g \in L^2$

$$\sum_{I\in D} a_I\, m_I\left(f v^{1/2}\right)\, m_I\left(g w^{1/2}\right) \;\le\; C\,Q\, \|f\|_{L^2}\, \|g\|_{L^2}\,. \tag{4.53}$$

Applying this theorem with $v = w^{-1}$ and $a_I = \left(K_I^+ + K_I^-\right)|I|^2$, we can see that the desired bound (4.49) will hold if we can prove the following three inequalities for every dyadic interval $J \in D$:

$$\frac{1}{|J|}\sum_{I\in D(J)} \left(K_I^+ + K_I^-\right)|I|^2\, m_I w\, m_I(w^{-1}) \;\le\; C[w]_{A_2^d}, \tag{4.54}$$

$$\frac{1}{|J|}\sum_{I\in D(J)} \left(K_I^+ + K_I^-\right)|I|^2\, m_I w \;\le\; C[w]_{A_2^d} m_J w, \tag{4.55}$$

$$\frac{1}{|J|}\sum_{I\in D(J)} \left(K_I^+ + K_I^-\right)|I|^2\, m_I(w^{-1}) \;\le\; C[w]_{A_2^d} m_J(w^{-1}). \tag{4.56}$$

It turns out that all these inequalities follow from the testing conditions (3.43). It is easy to see that inequality (4.54) follows from (3.43) and the definition of A_2-constant $\left(m_I w\, m_I(w^{-1}) \le [w]_{A_2^d}\right)$.

In order to see that inequalities (4.55) and (4.56) are true, we will need the following lemma, which can be found, for example, in [4]:

Lemma 6. *Let v be a weight, such that v^{-1} is a weight as well, and let $\{\lambda_I\}$ be a Carleson sequence of nonnegative numbers, that is, there exists a constant $Q > 0$ s.t.*

$$\forall J \in D \qquad \frac{1}{|J|}\sum_{I\in D(J)} \lambda_I \;\le\; Q,$$

then

$$\forall J \in D \qquad \frac{1}{|J|}\sum_{I\in D(J)} \frac{\lambda_I}{m_I(v^{-1})} \;\le\; 4Q\, m_J v$$

and, therefore, if $v \in A_2$ then for any $J \in D$ we have

$$\frac{1}{|J|}\sum_{I\in D(J)} m_I v\, \lambda_I \;\le\; 4Q\, \|v\|_{A_2^d}\, m_J v.$$

Applying this lemma to $\lambda_I = \left(K_I^+ + K_I^-\right)|I|^2$ and $v = w$, we obtain that (4.55) follows from (3.43). If we take $v = w^{-1}$ and observe that $\left\|w^{-1}\right\|_{A_2^d} = \|w\|_{A_2^d}$, we can see that (4.56) follows from (3.43) as well.

So,

$$\|T_1\|_{L^2(w)\to L^2(w)} \;\le\; C\, \|w\|_{A_2^d}\,.$$

Let us analyze operators T_3 and T_2 now.

$$T_3 f(x) \;=\; \sum_{I\in D} \left(K_I^+ - K_I^-\right) |I|^{3/2}\, m_I f\, h_I(x),$$

$$T_2 f(x) \;=\; \sum_{I\in D} \left(K_I^+ - K_I^-\right) \sqrt{|I|}\, \langle f; h_I\rangle\, \chi_I(x),$$

We first note that T_3 is a paraproduct operator and T_2 is its adjoint:

$$T_3 f(x) \;=\; \pi_b f(x) \;=\; \sum_{I\in D} m_I f\; b_I\; h_I(x),$$

$$T_2 f(x) \;=\; \pi_b^* f(x) \;=\; \sum_{I\in D} b_I\, \langle f; h_I\rangle\, \frac{1}{|I|}\chi_I(x)$$

with the sequence $b_I = \left(K_I^+ - K_I^-\right) |I|^{3/2}$.

In order for dyadic paraproduct and its adjoint to be bounded in L^2 we need b to be in the BMO^d, i.e. we need $\left\{b_I^2\right\}_{I\in D}$ to be a Carleson sequence, which is

$$\forall J \in D \qquad \frac{1}{|J|}\sum_{I\in D} \left(K_I^+ - K_I^-\right)^2 |I|^3 \;\leq\; Q$$

and coincides with $T(1)$ conditions (3.42).

It has been shown in [4] that the norm of the dyadic paraproduct (and, by symmetry of A_2-constant, of its adjoint as well) depends on the A_2-constant of the weight w at most linearly, i.e.

$$\|T_3\|_{L^2(w)\to L^2(w)} + \|T_2\|_{L^2(w)\to L^2(w)} \;\leq\; C\, Q^{1/2}[w]_{A_2^d}.$$

And, finally, the linear bound on the operator T_4,

$$T_4 f(x) \;=\; \sum_{I\in D} \left(K_I^+ + K_I^-\right) |I|\, \langle f; h_I\rangle\, h_I(x),$$

is implied by the uniform boundedness of the symbol (i.e. size conditions (3.41)) and Wittwer's weighted linear bound on the martingale transform (see [38]).

Since all parts of T obey linear bounds on their $L^2(w)$ norms with respect to the A_2-constant of the weight w, bound on the norm of T is linear as well,

$$\|T\|_{L^2(w)\to L^2(w)} \;\leq\; C\, \|w\|_{A_2}\,,$$

where constant C depends on the constant in the size condition (2.24), dyadic BMO^d-norms of $T(1)$ and $T^*(1)$ (at most linearly) and the constant in testing conditions (3.38) (at most linearly as well). Therefore, we have the following theorem.

Theorem 7. *Let T be a perfect dyadic operator on $\mathbb{R}$ such that*

$$\|T(1)\|_{BMO^d} \leqslant Q \quad \text{and} \quad \|T^*(1)\|_{BMO^d} \leqslant Q$$

and

$$\forall I \in D \quad \langle Th_I, h_I\rangle \leqslant Q$$

. Then T is bounded on $L^2(w)$ and

$$\|Tf\|_{L^2(w)} \leqslant CQ\|f\|_{L^2(w)},$$

with some constant C independent of the operator T.

Theorem 7 in the case $w(x) = 1$ is a $T(1)$ theorem, so we can view it as a weighted version of the $T(1)$ theorem. It is also interesting that the $L^2(w)$ norm of the operator T after proper normalization (we assume that the decay constant of the kernel is 1) only depends the BMO^d norms of $T(1)$ and $T^*(1)$ and the constant in testing conditions. It also depends on these constants at most linearly.

5. Two weight boundedness of perfect dyadic operators.

We start with the decomposition (3.44),

$$Tf(x) = \frac{1}{4}(T_1 f(x) + T_2 f(x) + T_3 f(x) + T_4 f(x)).$$

First we consider

$$T_1 f = \sum_{I\in D} (K_I^+ + K_I^-)|I| m_I f \chi_I(x).$$

Similarly to the one weight case, by duality using Theorem 5, we know it is bounded from $L^2(v)$ in $L^2(u)$ (with the norm bounded by CQ_1) whenever for every dyadic interval $J \in D$ the following three conditions hold simultaneously

$$\frac{1}{|J|} \sum_{I\in D(J)} (K_I^+ + K_I^-)|I|^2 m_I(v^{-1}) m_I u \leqslant Q_1, \tag{5.57}$$

$$\frac{1}{|J|} \sum_{I\in D(J)} (K_I^+ + K_I^-)|I|^2 m_I u \leqslant Q_1 m_J u, \tag{5.58}$$

$$\frac{1}{|J|} \sum_{I\in D(J)} (K_I^+ + K_I^-)|I|^2 m_I(v^{-1}) \leqslant Q_1 m_J(v^{-1}). \tag{5.59}$$

Second, consider the paraproduct T_3 and its adjoint T_2. In [5] we have sufficient conditions for the two weight boundedness for the dyadic paraproduct:

Theorem 8. *Let π_b be the dyadic paraproduct operator associated to the sequence $b = \{b_I\}_{I\in D}$:*

$$\pi_b f := \sum_{I\in D} m_I f\, b_I\, h_I.$$

Let (v, u) be a pair of measurable functions on $\mathbb{R}$ such that u and v^{-1}, the reciprocal of v, are weights on $\mathbb{R}$ and such that

(i) $(v, u) \in A_2^d$, that is $[v, u]_{A_2^d} := \sup_{I\in D} m_I(v^{-1})\, m_I u < \infty$.

(ii) Assume that there is a constant $\mathcal{C}_{v,u} > 0$ such that

$$\sum_{I\in D(J)} |\Delta_I u|^2 |I|\, m_I(v^{-1}) \leq \mathcal{C}_{v,u} u(J) \qquad \text{for all } J \in D,$$

where $\Delta_I u := m_{I_+} u - m_{I_-} u$, and $I_\pm$ are the right and left children of I.

Assume that $b \in Carl_{v,u}$, i.e. there is a constant $\mathcal{B}_{v,u} > 0$ such that

$$\sum_{I\in D(J)} \frac{|b_I|^2}{m_I u} \leq \mathcal{B}_{v,u} v^{-1}(J) \qquad \text{for all } J \in D.$$

Then π_b, the dyadic paraproduct associated to b, is bounded from $L^2(v)$ in $L^2(u)$. Moreover, there exist $C > 0$ such that

$$\|\pi_b f\|_{L^2(u)} \le C\sqrt{[v,u]_{A_2^d}\mathcal{B}_{v,u}}\Big(\sqrt{[v,u]_{A_2^d}} + \sqrt{\mathcal{C}_{v,u}}\Big)\|f\|_{L^2(v)}\,.$$

Therefore, in order to be able to bound the paraproduct T_3 that has symbol $b_I = (K_I^+ - K_I^-)|I|^{3/2}$, we need the following three conditions to hold:

$$[v,u]_{A_2^d} := \sup_{I\in D} m_I(v^{-1})\, m_I u < \infty, \tag{5.60}$$

$$\sum_{I\in D(J)} |\Delta_I u|^2 |I|\, m_I(v^{-1}) \le \mathcal{C}_{v,u} u(J) \qquad \text{for all } J \in D, \tag{5.61}$$

$$\sum_{I\in D(J)} \frac{(K_I^+ - K_I^-)^2|I|^3}{m_I u} \le \mathcal{B}_{v,u} v^{-1}(J) \qquad \text{for all } J \in D. \tag{5.62}$$

The operator T_2 is adjoint to T_3, $T_2 = T_3^*$, so T_2 is bounded from $L^2(v)$ in $L^2(u)$ whenever the paraproduct T_3 is bounded from $L^2(u^{-1})$ in $L^2(v^{-1})$. Therefore, in order for T_2 to be bounded from $L^2(v)$ in $L^2(u)$ we need the following three conditions to hold:

$$[u^{-1},v^{-1}]_{A_2^d} := \sup_{I\in D} m_I(u)\, m_I(v^{-1}) < \infty, \tag{5.63}$$

$$\sum_{I\in D(J)} |\Delta_I(v^{-1})|^2 |I|\, m_I u \le \mathcal{C}_{u^{-1},v^{-1}} v^{-1}(J) \qquad \text{for all } J \in D, \tag{5.64}$$

$$\sum_{I\in D(J)} \frac{(K_I^+ - K_I^-)^2|I|^3}{m_I(v^{-1})} \le \mathcal{B}_{u^{-1},v^{-1}} u(J) \qquad \text{for all } J \in D. \tag{5.65}$$

Finally, consider T_4, the martingale transform with symbol $\sigma = (K_I^+ + K_I^-)|I|$, which is uniformly bounded by (3.41),

$$T_4 f(x) \;=\; \sum_{I\in D} \big(K_I^+ + K_I^-\big)\,|I|\,\langle f; h_I\rangle\, h_I(x).$$

Necessary and sufficient conditions were obtained in [28]:

Theorem 9. *Let $\sigma = \{\sigma_I\}_{I\in D}$ be a sequence of signs $\pm$. The family of operators T_σ of martingale transforms with symbol σ is uniformly bounded from $L^2(v)$ to $L^2(u)$ if and only if the following assertions hold simultaneously:*

$$\forall J \in D \quad m_J u\, m_J v^{-1} \leqslant C < \infty, \tag{5.66}$$

$$\forall J \in D \;\; \frac{1}{|J|}\sum_{I\in D(J)} (\Delta_I u)^2 m_I(v^{-1})|I| \leqslant C m_I u, \tag{5.67}$$

$$\forall J \in D \;\; \frac{1}{|J|}\sum_{I\in D(J)} (\Delta_I(v^{-1}))^2 m_I u |I| \leqslant C m_I(v^{-1}), \tag{5.68}$$

and an operator T_0*, defined by* $T_0 f := \sum_{I\in D}\left|\frac{\Delta_I(v^{-1})\Delta_I u}{m_I(v^{-1})m_I u}\right| m_I f \chi_I(x)$*, is bounded from* $L^2(v)$ *to* $L^2(u)$*, or, equivalently*

$$\forall J\in D\ \ \frac{1}{|J|}\int_J\left(\sum_{I\in D(J)}\left|\frac{\Delta_I u\Delta_I(v^{-1})}{m_I u}\right|\chi_I(x)\right)^2 u dx \leqslant C m_J(v^{-1}) \tag{5.69}$$

and

$$\forall J\in D\ \ \frac{1}{|J|}\int_J\left(\sum_{I\in D(J)}\left|\frac{\Delta_I u\Delta_I(v^{-1})}{m_I (v^{-1})}\right|\chi_I(x)\right)^2 v^{-1} dx \leqslant C m_J u. \tag{5.70}$$

Now we need to put all conditions we obtained for the operators $T_{1,2,3,4}$, conditions (5.57), (5.58), (5.59), (5.60), (5.61), (5.62), (5.63), (5.64), (5.65), (5.66)(5.67)(5.68)(5.69)(5.70), together. Note that (5.60), (5.63) and (5.66) are all the same joint dyadic A_2^d condition, while condition (5.57) follows from the joint A_2^d and (3.43) under no additional assumptions. Note also that conditions (5.61) and (5.67) are the same, conditions (5.64) and (5.68) also coincide. Therefore, we obtain the following theorem.

Theorem 10. *Let* (v,u) *be a pair of measurable functions, such that* u *and* v^{-1}*, the reciprocal of* v*, are weights on* $\mathbb{R}$ *such that*

$$(v,u)\in A_2^d\,,\ i.e.\ \ [v,u]_{A_2^d} := \sup_{I\in D} m_I(v^{-1}) m_I u < \infty, \tag{5.71}$$

$$\forall J\in D\ \ \frac{1}{|J|}\sum_{I\in D(J)} (\Delta_I u)^2 m_I(v^{-1})|I| \leqslant C m_I u, \tag{5.72}$$

$$\forall J\in D\ \ \frac{1}{|J|}\sum_{I\in D(J)} (\Delta_I (v^{-1}))^2 m_I u|I| \leqslant C m_I(v^{-1}), \tag{5.73}$$

and operator T_0 *is bounded from* $L^2(v)$ *in* $L^2(u)$*, i.e.*

$$\forall J\in D\ \ \frac{1}{|J|}\int_J\left(\sum_{I\in D(J)}\left|\frac{\Delta_I u\Delta_I(v^{-1})}{m_I u}\right|\chi_I(x)\right)^2 u(x) dx \leqslant C m_J(v^{-1}) \tag{5.74}$$

and

$$\forall J\in D\ \ \frac{1}{|J|}\int_J\left(\sum_{I\in D(J)}\left|\frac{\Delta_I u\Delta_I(v^{-1})}{m_I (v^{-1})}\right|\chi_I(x)\right)^2 v^{-1}(x) dx \leqslant C m_J u. \tag{5.75}$$

Let T *be a perfect dyadic operator with* $\|T(1)\|_{BMO^d}, \|T^*(1)\|_{BMO^d}\leqslant Q$ *and for every* $I\in D$ *we have* $\langle Th_I; h_I\rangle \leqslant Q$*.*

Then T *is bounded from* $L^2(v)$ *to* $L^2(u)$ *whenever for any dyadic interval* $J\in D$

$$\frac{1}{|J|}\sum_{I\in D(J)} (K_I^+ + K_I^-)|I|^2 m_I u \leqslant C m_J u, \tag{5.76}$$

$$\frac{1}{|J|}\sum_{I\in D(J)}(K_I^+ + K_I^-)|I|^2 m_I(v^{-1}) \leqslant C m_J(v^{-1}), \tag{5.77}$$

$$\frac{1}{|J|}\sum_{I\in D(J)}\frac{(K_I^+ - K_I^-)^2|I|^3}{m_I u} \leq C m_J(v^{-1}) \tag{5.78}$$

and

$$\frac{1}{|J|}\sum_{I\in D(J)}\frac{(K_I^+ - K_I^-)^2|I|^3}{m_I(v^{-1})} \leq C m_J u. \tag{5.79}$$

6. Sufficient conditions for the two weight boundedness of perfect dyadic operators under additional assumptions that weighted are in A_∞^d

In this section we will assume that v^{-1} and u are A_∞^d weights and show that conditions of Theorem 10 can be reduced in this case. A weight w belongs to the class A_∞^d whenever

$$[w]_{A_\infty^d} := \sup_{I\in D} m_I w e^{-m_I(\ln w)} \;<\; \infty.$$

From [Be1] we have the following lemma

Lemma 11. *Let w be a weight, such that $\log w \in L^1_{loc}$, let $\{\lambda_I\}_{I\in D}$ be a Carleson sequence:*

$$\exists Q>0 \;\; s.t. \;\; \forall J\in D \quad \frac{1}{|J|}\sum_{I\in D(J)}\lambda_I \leqslant Q.$$

Then for every dyadic interval $J\in D$

$$\frac{1}{|J|}\sum_{I\in D(J)} e^{m_I(\log w)}\lambda_I \leqslant 4Q m_J w$$

and therefore, if $w\in A_\infty^d$ then for every dyadic interval $J\in D$ we have:

$$\frac{1}{|J|}\sum_{I\in D(J)} m_I w\,\lambda_I \leqslant 4Q[w]_{A_\infty^d} m_I w.$$

In particular, $u, v^{-1}\in A_\infty^d$ implies that u and v^{-1} are in RH_1^d (see [6]), where RH_1^d is defined as follows

$$w\in RH_1^d \quad\Longleftrightarrow\quad [w]_{RH_1^d} := \sup_{I\in D} m_I\left(\frac{w}{m_I w}\log\frac{w}{m_I w}\right) \;<\; \infty.$$

We also know from [6] that $[w]_{RH_1^d} \leqslant \log 16[w]_{A_\infty^d}$ and we have the following theorem that first appeared in [7] without the sharp constant and in [6] it was stated in stronger form and the constant was traced. Here we state the strong form from [6].

Theorem 12. *Let w be almost everywhere positive locally integrable function on the real line and J be any interval. Then up to a numerical constant*

$$m_J(w\log w) - m_J w\log m_J w \approx \frac{1}{|J|}\sum_{I\in D(J)}\left(\frac{\Delta_I w}{m_I w}\right)^2 m_I w|I|.$$

In particular case when w is a weight in RH_1^d, we have that for every dyadic interval $J \in D$

$$\frac{1}{|J|}\sum_{I\in D(J)}\left(\frac{\Delta_I w}{m_I w}\right)^2 m_I w|I| \leqslant C[w]_{RH_1^d} m_J w. \tag{6.80}$$

Firstly, note that conditions (5.72) and (5.73) follow from the fact that u and v^{-1} are in A_∞^d, Theorem 12 and the joint A_2^d condition (6.81).

Secondly, conditions (5.76) and (5.77) by Lemma 11 follow from the fact that $u, v^{-1} \in A_\infty^d$ and 3.43. Therefore, under additional assumptions that $u, v^{-1} \in A_\infty^d$, we can simplify Theorem 10 as follows.

Theorem 13. *Let (v,u) be a pair of measurable functions, such that u and v^{-1}, the reciprocal of v, are A_∞^d weights on $\mathbb{R}$ such that*

$$(v,u) \in A_2^d\,, \text{i.e. } [v,u]_{A_2^d} := \sup_{I\in D} m_I(v^{-1}) m_I u < \infty, \tag{6.81}$$

and operator T_0 is bounded from $L^2(v)$ in $L^2(u)$, i.e.

$$\forall J \in D \quad \frac{1}{|J|}\int_J \left(\sum_{I\in D(J)}\left|\frac{\Delta_I u \Delta_I(v^{-1})}{m_I u}\right|\chi_I(x)\right)^2 u(x)dx \leqslant C m_J(v^{-1}) \tag{6.82}$$

and

$$\forall J \in D \quad \frac{1}{|J|}\int_J \left(\sum_{I\in D(J)}\left|\frac{\Delta_I u \Delta_I(v^{-1})}{m_I(v^{-1})}\right|\chi_I(x)\right)^2 v^{-1}(x)dx \leqslant C m_J u. \tag{6.83}$$

Let T be a perfect dyadic operator with $\|T(1)\|_{BMO^d}, \|T^*(1)\|_{BMO^d} \leqslant Q$ and for every $I \in D$ we have $\langle Th_I; h_I\rangle \leqslant Q$.

Then T is bounded from $L^2(v)$ to $L^2(u)$ whenever for any dyadic interval $J \in D$

$$\frac{1}{|J|}\sum_{I\in D(J)}\frac{(K_I^+ - K_I^-)^2|I|^3}{m_I u} \le C m_J(v^{-1}) \tag{6.84}$$

and

$$\frac{1}{|J|}\sum_{I\in D(J)}\frac{(K_I^+ - K_I^-)^2|I|^3}{m_I(v^{-1})} \le C m_J u. \tag{6.85}$$

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Oleksandra Beznosova, Department of Mathematics, University of Alabama, 345 Gordon Palmer Hall, Tuscaloosa, AL 35487

E-mail address: `ovbeznosova@ua.edu`