

# SHARP BOUNDS FOR $T$ -HAAR MULTIPLIERS ON $L^2$

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ABSTRACT. We show that if a weight  $w \in C_{2t}^d$  and there is  $q > 1$  such that  $w^{2t} \in A_q^d$ , then the  $L^2$ -norm of the  $t$ -Haar multiplier of complexity  $(m, n)$  associated to  $w$  depends on the square root of the  $C_{2t}^d$ -characteristic of  $w$  times the square root  $A_q^d$ -characteristic of  $w^{2t}$  times a constant that depends polynomially on the complexity. In particular, if  $w \in C_{2t}^d \cap A_\infty^d$  then  $w^{2t} \in A_q^d$  for some  $q > 1$ .

## 1. INTRODUCTION

Recently Tuomas Hytönen settled the  $A_2$ -conjecture [H]: for all Calderón-Zygmund integral singular operators  $T$  in  $\mathbb{R}^N$ , weights  $w \in A_p$ , there is  $C_{p,N,T} > 0$  such that,

$$\|Tf\|_{L^p(w)} \leq C_{p,N,T} [w]_{A_p}^{\max\{1, \frac{1}{p-1}\}} \|f\|_{L^p(w)}.$$

In his proof he developed and used a representation valid for *any* Calderón-Zygmund operator as an average of Haar shift operators of *arbitrary complexity*, paraproducts and their adjoints. See [L1, P4] for surveys of the  $A_2$ -conjecture. An important and hard part of the proof was to obtain bounds for Haar shifts operators that depended linearly in the  $A_2$ -characteristic and at most polynomially in the complexity.

In this paper we show that if a weight  $w \in C_{2t}^d \cap A_\infty^d$ , then the  $L^2$ -norm of the  $t$ -Haar multiplier of complexity  $(m, n)$  associated to  $w$  depends on the square root of the  $C_{2t}^d$ -characteristic of  $w$  times the square root  $A_q^d$ -characteristic of  $w^{2t}$  for some  $q > 1$  depending on  $t \in \mathbb{R}$  times a constant that depends polynomially on the complexity.

For  $t \in \mathbb{R}$ ,  $m, n \in \mathbb{N}$ , and a weight  $w$ , the  $t$ -Haar multiplier of complexity  $(m, n)$  was introduced in [MoP], and is defined formally by

$$T_{t,w}^{m,n} f(x) = \sum_{L \in \mathcal{D}} \sum_{I \in \mathcal{D}_m(L), J \in \mathcal{D}_n(L)} \frac{\sqrt{|I||J|}}{|L|} \frac{w^t(x)}{(m_L w)^t} \langle f, h_I \rangle h_J(x),$$

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where  $\mathcal{D}$  denotes the dyadic intervals,  $|I|$  the length of interval  $I$ ,  $\mathcal{D}_m(L)$  denotes the dyadic subintervals of  $L$  of length  $2^{-m}|L|$ ,  $h_I$  is a Haar function associated to  $|I|$ , and  $\langle f, g \rangle$  denotes the  $L^2$ -inner product.

When  $(m, n) = (0, 0)$  we denote the corresponding Haar multiplier by  $T_w^t$ , and, if in addition  $t = 1$ , simply  $T_w$ . The Haar multipliers  $T_w$  are closely related to the resolvent of the dyadic paraproduct [P1], and appeared in the study of Sobolev spaces on Lipschitz curves [P3].

A necessary condition for the boundedness of  $T_{w,t}^{m,n}$  on  $L^2(\mathbb{R}^N)$  is that  $w \in C_{2t}^d$ , see [MoP], that is,

$$[w]_{C_{2t}^d} := \sup_{I \in \mathcal{D}} \left( \frac{1}{|I|} \int_I w^{2t}(x) dx \right) \left( \frac{1}{|I|} \int_I w(x) dx \right)^{-2t} < \infty.$$

This condition is also sufficient for  $t < 0$  and  $t \geq 1/2$ . Notice that for  $0 \leq t < 1/2$  the condition  $C_{2t}^d$  is always fulfilled; in this case, boundedness of  $T_{w,t}^{m,n}$  is known when  $w \in A_\infty^d$  [MoP, KP]. The first author showed in [Be, Chapter 5], that if  $w \in C_{2t}^d$  and  $w^{2t} \in A_q^d$  then the  $L^2$ -norm of  $T_w^t$ , is bounded by a constant times  $[w]_{C_{2t}^d}^{\frac{1}{2}} [w^{2t}]_{A_q^d}^{\frac{1}{2}}$ . Here we present a different proof of this result that holds for  $t$ -Haar multipliers of complexity  $(m, n)$  with polynomial dependence on the complexity.

**Theorem 1.1.** *Let  $w \in C_{2t}^d$  and assume there is  $q > 1$  such that  $w^{2t} \in A_q^d$ , then there is a constant  $C_q > 0$  depending only on  $q$ , such that*

$$\|T_{t,w}^{m,n} f\|_2 \leq C_q (m+n+2)^3 [w]_{C_{2t}^d}^{\frac{1}{2}} [w^{2t}]_{A_q^d}^{\frac{1}{2}} \|f\|_2.$$

When  $w^{2t} \in A_2^d$ , this was proved in [MoP].

Using known properties of weights we can replace the condition  $w^{2t} \in A_q^d$ , by what may seem to be a more natural condition  $w \in C_{2t}^d \cap A_\infty^d$ .

**Theorem 1.2.** *Let  $w \in C_{2t}^d \cap A_\infty^d$ , then*

- (i) *if  $0 \leq 2t < 1$ , there is  $q > 1$  such that  $w \in A_q^d$ , then  $w^{2t} \in A_q^d$ , and*

$$\|T_{t,w}^{m,n} f\|_2 \leq C_q (m+n+2)^3 [w^{2t}]_{A_q^d}^{\frac{1}{2}} \|f\|_2 \leq C_q (m+n+2)^3 [w]_{A_q^d}^t \|f\|_2.$$

- (ii) *If  $2t \geq 1$  and  $w \in A_p^d$  then for  $q = 2t(p-1) + 1$ ,  $w^{2t} \in A_q^d$ , and*

$$\|T_{t,w}^{m,n} f\|_2 \leq C_q (m+n+2)^3 [w]_{C_{2t}^d}^{\frac{1}{2}} [w^{2t}]_{A_q^d}^{\frac{1}{2}} \|f\|_2 \leq C_p (m+n+2)^3 [w]_{C_{2t}^d} [w]_{A_p^d} \|f\|_2.$$

- (iii) *If  $t < 0$  then for  $q = 1 - 2t$ ,  $w^{2t} \in A_q^d$ , and the bound becomes linear in the  $C_{2t}^d$  characteristic of  $w$ ,*

$$\|T_{t,w}^{m,n} f\|_2 \leq C (m+n+2)^3 [w]_{C_{2t}^d} \|f\|_2.$$

The result was known to be optimal when  $t = \pm 1/2$  [Be, P2]. The bound in (ii) is not optimal since for  $t = 1$ , the  $L^2$  norm of  $T_w$  is bounded by a constant times  $[w]_{C_2^d} D(w)$ , where  $D(w)$  is the doubling constant of  $w$ , see [P2]. Here we get the larger norm  $C[w]_{C_2^d}[w]_{A_p^d}$ .

To prove this theorem we modify the argument in [MoP] that works when  $w \in A_2^d$  ( $p = 2$ ). In particular we need a couple of new  $A_p$ -weight lemmas that are proved using Bellman function techniques: the  $A_p$ -Little Lemma, and the  $\alpha\beta$ -Lemma.

A few open questions remain. In case (i)  $0 < 2t < 1$ , is  $w^{2t} \in A_\infty^d$  a necessary condition for the boundedness of  $T_{tw}^{m,n}$ ? Here we show is sufficient. Is it possible to get an estimate independent of  $q > 1$  such that  $w^{2t} \in A_q^d$ ? More specifically, can we replace  $C_q[w^{2t}]_{A_q^d}^{1/2}$  by  $C[w^{2t}]_{A_\infty^d}$ ? or even better by  $CD(w)$ ? Similarly in case (ii).

The paper is organized as follows. In Section 2 we provide the basic definitions and basic results that are used through-out this paper. In Section 3 we prove the lemmas that are essential for the main result. In Section 4 we prove the main estimate for the  $t$ -Haar multipliers with complexity  $(m, n)$ . In the Appendix we prove the  $A_p$ -Little Lemma.

## 2. PRELIMINARIES

**2.1. Weights, maximal function and dyadic intervals.** A *weight*  $w$  is a locally integrable function in  $\mathbb{R}$  positive almost everywhere. The  $w$ -measure of a measurable set  $E$ , denoted by  $w(E)$ , is  $w(E) = \int_E w(x)dx$ . For a measure  $\sigma$ ,  $\sigma(E) = \int_E d\sigma$ , and  $|E|$  stands for the Lebesgue measure of  $E$ . We define  $m_E^\sigma f$  to be the integral average of  $f$  on  $E$ , with respect to  $\sigma$ ,

$$m_E^\sigma f := \frac{1}{\sigma(E)} \int_E f(x) d\sigma.$$

When  $dx = d\sigma$  we simply write  $m_E f$ , when  $d\sigma = v dx$  we write  $m_E^v f$ .

Given a weight  $w$ , a measurable function  $f : \mathbb{R}^N \rightarrow \mathbb{C}$  is in  $L^p(w)$  if and only if  $\|f\|_{L^p(w)} := \left( \int_{\mathbb{R}^N} |f(x)|^p w(x) dx \right)^{1/p} < \infty$ .

For a weight  $v$  we define the *weighted maximal function of  $f$*  by

$$(M_v f)(x) = \sup_{I:x \in I} m_I^v |f|$$

where  $I$  is a cube in  $\mathbb{R}^N$  with sides parallel to the axis. The operator  $M_v$  is bounded in  $L^p(v)$  for all  $p > 1$  and furthermore

$$(2.1) \quad \|M_v f\|_{L^p(v)} \leq Cp' \|f\|_{L^q(v)},$$

where  $p'$  is the dual exponent of  $p$ , that is  $1/p + 1/p' = 1$ . A proof of this fact can be found in [CrMPz1]. When  $v = 1$ ,  $M_v$  is the usual

Hardy-Littlewood maximal function, which we will denote by  $M$ . It is well-known that  $M$  is bounded in  $L^p(w)$  if and only if  $w \in A_p$  [Mu].

The collection of all *dyadic intervals*,  $\mathcal{D}$ , is given by:  $\mathcal{D} = \cup_{n \in \mathbb{Z}} \mathcal{D}_n$ , where  $\mathcal{D}_n := \{I \subset \mathbb{R} : I = [k2^{-n}, (k+1)2^{-n}), k \in \mathbb{Z}\}$ . For a dyadic interval  $L$ , let  $\mathcal{D}(L)$  be the collection of its dyadic subintervals,  $\mathcal{D}(L) := \{I \subset L : I \in \mathcal{D}\}$ , and let  $\mathcal{D}_n(L)$  be the  $n^{\text{th}}$ -generation of dyadic subintervals of  $L$ ,  $\mathcal{D}_n(L) := \{I \in \mathcal{D}(L) : |I| = 2^{-n}|L|\}$ .

For every dyadic interval  $I \in \mathcal{D}_n$  there is exactly one  $\widehat{I} \in \mathcal{D}_{n-1}$ , such that  $I \subset \widehat{I}$ ,  $\widehat{I}$  is called the parent of  $I$ . Each dyadic interval  $I$  in  $\mathcal{D}_n$  has two children in  $\mathcal{D}_{n+1}$ , the right and left halves, denoted  $I_+$  and  $I_-$  respectively.

A weight  $w$  is *dyadic doubling* if  $w(\widehat{I})/w(I) \leq C$  for all  $I \in \mathcal{D}$ . The smallest constant  $C$  is called the doubling constant of  $w$  and is denoted by  $D(w)$ . Note that  $D(w) \geq 2$ , and that in fact the ratio between the length of a child and the length of its parent is comparable to one, more precisely,  $D(w)^{-1} \leq w(I)/w(\widehat{I}) \leq 1 - D(w)^{-1}$ .

**2.2. Dyadic  $A_p^d$ , reverse Hölder  $RH_p^d$  and  $C_s^d$  classes.** A weight  $w$  is said to belong to the *dyadic Muckenhoupt  $A_p^d$ -class* if and only if

$$[w]_{A_p^d} := \sup_{I \in \mathcal{D}} (m_I w)(m_I w^{\frac{-1}{p-1}})^{p-1} < \infty, \quad \text{for } 1 < p < \infty,$$

where  $[w]_{A_p^d}$  is called the  $A_p^d$ -characteristic of the weight. If a weight is in  $A_p^d$  then it is dyadic doubling. These classes are nested,  $A_p^d \subset A_q^d$  for all  $p \leq q$ . The class  $A_\infty^d$  is defined by  $A_\infty^d := \bigcup_{p>1} A_p^d$ .

A weight  $w$  is said to belong to the *dyadic reverse Hölder  $RH_p^d$ -class* if and only if

$$[w]_{RH_p^d} := \sup_{I \in \mathcal{D}} (m_I w^p)^{\frac{1}{p}} (m_I w)^{-1} < \infty, \quad \text{for } 1 < p < \infty,$$

where  $[w]_{RH_p^d}$  is called the  $RH_p^d$ -characteristic of the weight. If a weight is in  $RH_p^d$  then it is not necessarily dyadic doubling (in the non-dyadic setting reverse Hölder weights are always doubling). Also these classes are nested,  $RH_p^d \subset RH_q^d$  for all  $p \geq q$ . The class  $RH_1^d$  is defined by  $RH_1^d := \bigcup_{p>1} RH_p^d$ . In the non-dyadic setting  $A_\infty = RH_1$ . In the dyadic setting the collection of dyadic doubling weights in  $RH_1^d$  is  $A_\infty^d$ , hence  $A_\infty^d$  is a proper subset of  $RH_1^d$ . See [BeRez] for some recent and very interesting results relating these classes.

The following are well-known properties of weights (see [JN]) for (ii):

**Lemma 2.1.** *The following hold*

- If  $0 \leq s \leq 1$  and  $w \in A_\infty^d$  then  $w^s \in A_\infty$ . More precisely, if  $p > 1$  and  $w \in A_p^d$  then  $w^s \in A_p$ , and  $[w^s]_{A_p^d} \leq [w]_{A_p^d}^s$ .
- If  $s, q > 1$  then  $w \in RH_s^d \cap A_q^d$  if and only if  $w^s \in A_{s(q-1)+1}$ . Moreover  $[w^s]_{A_{s(q-1)+1}} \leq [w]_{RH_s^d}^s [w]_{A_q^d}^s$ ,  $[w]_{A_q^d}^s \leq [w^s]_{A_{s(q-1)+1}}$ , and  $[w]_{RH_s^d}^s \leq [w^s]_{A_{s(q-1)+1}}$ .
- If  $p > 1$ , and  $1/p + 1/p' = 1$ , then  $w \in A_p^d$  if and only if  $w^{-1/p-1} \in A_{p'}$ . Moreover  $[w]_{A_p^d} = [w^{-1/p-1}]_{A_{p'}^d}^{p-1}$ .

The following property can be found in [GaRu],

**Lemma 2.2.** *If  $w \in RH_s^d \cap A_q^d$  then for all  $E \subset B$ ,*

$$(|E|/|B|)^q [w]_{A_q^d}^{-1} \leq w(E)/w(B) \leq (|E|/|B|)^{1-\frac{1}{s}} [w]_{RH_s^d}.$$

*In particular  $D(w) \leq 2^q [w]_{A_q^d}$ .*

A weight  $w$  satisfies the  $C_s^d$ -condition, for  $s \in \mathbb{R}$ , if

$$[w]_{C_s^d} := \sup_{I \in \mathcal{D}} (m_I w^s) (m_I w)^{-s} < \infty.$$

The quantity defined above is called the  $C_s^d$ -characteristic of  $w$ . The class of weights  $C_s^d$  was defined in [KP]. Let us analyze this definition. For  $0 \leq s \leq 1$ , we have that any weight satisfies the condition with  $C_s^d$ -characteristic 1, this is just a consequence of Hölder's Inequality (for  $s = 0, 1$  is trivial). When  $s > 1$ , the condition is analogous to the dyadic reverse Hölder condition and  $[w]_{C_s^d}^{1/s} = [w]_{RH_s^d}$ . For  $s < 0$ , we have that  $w \in C_s^d$  if and only if  $w \in A_{1-1/s}^d$ , moreover  $[w]_{C_s^d} = [w]_{A_{1-1/s}^d}^{-s}$ .

**Lemma 2.3.** *If  $w \in C_s^d \cap A_\infty^d$  then the following hold*

- For all  $0 \leq s \leq 1$ , there is a  $p > 1$  such that  $w^s \in A_p$ .
- If  $s > 1$  then there is  $q > 1$  such that  $w^s \in A_{s(q-1)+1}$ .
- If  $s < 0$  then  $w^s \in A_{1-s}$ .

The proof of this lemma is a direct application of Lemma 2.1 item by item.

**2.3. Weighted Haar functions.** For a given weight  $v$  and an interval  $I$  define the *weighted Haar function* as

$$(2.2) \quad h_I^v(x) = \frac{1}{v(I)} \left( \sqrt{\frac{v(I_-)}{v(I_+)}} \chi_{I_+}(x) - \sqrt{\frac{v(I_+)}{v(I_-)}} \chi_{I_-}(x) \right),$$

where  $\chi_I(x)$  is the characteristic function of the interval  $I$ .

If  $v$  is the Lebesgue measure on  $\mathbb{R}$ , we will denote the *Haar function* simply by  $h_I$ . It is a simple exercise to verify that the weighted and unweighted Haar functions are related linearly as follows,

*Proposition 2.4.* For any weight  $v$ , there are numbers  $\alpha_I^v, \beta_I^v$  such that

$$h_I(x) = \alpha_I^v h_I^v(x) + \beta_I^v \chi_I(x) / \sqrt{|I|}$$

where (i)  $|\alpha_I^v| \leq \sqrt{m_I v}$ , (ii)  $|\beta_I^v| \leq |\Delta_I v| / m_I v$ ,  $\Delta_I v := m_{I^+} v - m_{I^-} v$ .

The family  $\{h_I^v\}_{I \in \mathcal{D}}$  is an orthonormal system in  $L^2(v)$ , with inner product  $\langle f, g \rangle_v := \int_{\mathbb{R}} f(x) \overline{g(x)} v(x) dx$ .

**2.4. Carleson sequences.** If  $v$  is a weight, a positive sequence  $\{\alpha_I\}_{I \in \mathcal{D}}$  is called a  *$v$ -Carleson sequence with intensity  $B$*  if for all  $J \in \mathcal{D}$ ,

$$(2.3) \quad \frac{1}{|J|} \sum_{I \in \mathcal{D}(J)} \lambda_I \leq B m_J v.$$

When  $v = 1$  we call a sequence satisfying (2.3) for all  $J \in \mathcal{D}$  a *Carleson sequence with intensity  $B$* .

*Proposition 2.5.* Let  $v$  be a weight,  $\{\lambda_I\}_{I \in \mathcal{D}}$  and  $\{\gamma_I\}_{I \in \mathcal{D}}$  be two  $v$ -Carleson sequences with intensities  $A$  and  $B$  respectively then for any  $c, d > 0$  we have that

- (i)  $\{c\lambda_I + d\gamma_I\}_{I \in \mathcal{D}}$  is a  $v$ -Carleson sequence with intensity  $cA + dB$ .
- (ii)  $\{\sqrt{\lambda_I} \sqrt{\gamma_I}\}_{I \in \mathcal{D}}$  is a  $v$ -Carleson sequence with intensity  $\sqrt{AB}$ .
- (iii)  $\{(c\sqrt{\lambda_I} + d\sqrt{\gamma_I})^2\}_{I \in \mathcal{D}}$  is a  $v$ -Carleson sequence with intensity  $2c^2A + 2d^2B$ .

The proof of these statements is quite simple, see [MoP].

### 3. MAIN TOOLS

In this section, we state the lemmas and theorems necessary to get the estimate for the  $t$ -Haar multipliers of complexity  $(m, n)$ .

**3.1. Carleson Lemmas.** The Weighted Carleson Lemma we present here is a variation in the spirit of other weighted Carleson embedding theorems that appeared before in the literature [NV, NTV1]. You can find a proof in [MoP].

**Lemma 3.1** (Weighted Carleson Lemma). *Let  $v$  be a weight, then  $\{\alpha_L\}_{L \in \mathcal{D}}$  is a  $v$ -Carleson sequence with intensity  $B$  if and only if for all non-negative  $v$ -measurable functions  $F$  on the line,*

$$(3.1) \quad \sum_{L \in \mathcal{D}} \alpha_L \inf_{x \in L} F(x) \leq B \int_{\mathbb{R}} F(x) v(x) dx.$$

The following lemma we view as a finer replacement for Hölder's inequality:  $1 \leq (m_I w)(m_I w^{-1/(p-1)})^{p-1}$ .

**Lemma 3.2** ( $A_p$ -Little Lemma). *Let  $v$  be a weight, such that  $v^{-1/(p-1)}$  is a weight as well, and let  $\{\lambda_I\}_{I \in \mathcal{D}}$  be a Carleson sequence with intensity  $Q$  then  $\{\lambda_I/(m_I v^{-1/(p-1)})^{p-1}\}_{I \in \mathcal{D}}$  is a  $v$ -Carleson sequence with intensity  $4Q$ , that is for all  $J \in \mathcal{D}$ ,*

$$\frac{1}{|J|} \sum_{I \in \mathcal{D}(J)} \frac{\lambda_I}{(m_I v^{-1/(p-1)})^{p-1}} \leq 4Q m_J v.$$

For  $p = 2$  this was proved in [Be, Proposition 3.4], or [Be1, Proposition 2.1], using the same Bellman function as in the proof we present in the Appendix.

**Lemma 3.3** ([NV]). *Let  $v$  be a weight such that  $v^{-1/(p-1)}$  is also a weight. Let  $\{\lambda_J\}_{J \in \mathcal{D}}$  be a Carleson sequence with intensity  $B$ . Let  $F$  be a non-negative measurable function on the line. Then*

$$\sum_{J \in \mathcal{D}} \frac{\lambda_J}{(m_J v^{-1/(p-1)})^{p-1}} \inf_{x \in J} F(x) \leq C B \int_{\mathbb{R}} F(x) v(x) dx.$$

Lemma 3.3 is an immediate consequence of Lemma 3.2, and the Weighted Carleson Lemma 3.1. Note that Lemma 3.2 can be deduced from Lemma 3.3 with  $F(x) = \chi_J(x)$ .

The following lemma, for  $v = w^{-1}$ , and for  $\alpha = 1/4$  appeared in [Be], and for  $0 < \alpha < 1/2$ , in [NV]. With small modification in their proof, using the Bellman function  $B(x, y) = x^\alpha y^\beta$  with domain of definition the first quadrant  $x, y > 0$ , we can accomplish the result below, for a complete proof see [Mo].

**Lemma 3.4.** ( $\alpha\beta$ -Lemma) *Let  $u, v$  be weights. Then for any  $J \in \mathcal{D}$  and any  $\alpha, \beta \in (0, 1/2)$*

$$(3.2) \quad \frac{1}{|J|} \sum_{I \in \mathcal{D}(J)} \frac{|\Delta_I u|^2}{(m_I u)^2} |I| (m_I u)^\alpha (m_I v)^\beta \leq C_{\alpha, \beta} (m_J u)^\alpha (m_J v)^\beta.$$

The constant  $C_{\alpha, \beta} = 36/\min\{\alpha - 2\alpha^2, \beta - 2\beta^2\}$ .

From this lemma we immediately deduce the following,

**Lemma 3.5.** *Let  $1 < q < \infty$ ,  $w \in A_q^d$ , then  $\{\mu_I^{q, \alpha}\}_{I \in \mathcal{D}}$ , where*

$$\mu_I^{q, \alpha} := (m_I w)^\alpha (m_I w^{\frac{-1}{q-1}})^{\alpha(q-1)} |I| \left( \frac{|\Delta_I w|^2}{(m_I w)^2} + \frac{|\Delta_I w^{\frac{-1}{q-1}}|^2}{(m_I w^{\frac{-1}{q-1}})^2} \right),$$

is a Carleson sequence with Carleson intensity at most  $C_\alpha[w]_{A_q}^\alpha$  for any  $\alpha \in (0, \max\{1/2, 1/2(q-1)\})$ . Moreover,  $\{\nu_I^q\}_{I \in \mathcal{D}}$ , where

$$\nu_I^q := (m_I w)(m_I w^{\frac{-1}{q-1}})^{(q-1)} |I| \left( \frac{|\Delta_I w|^2}{(m_I w)^2} + \frac{|\Delta_I w^{\frac{-1}{q-1}}|^2}{(m_I w^{\frac{-1}{q-1}})^2} \right)$$

is a Carleson sequence with Carleson intensity at most  $C[w]_{A_q}$ .

*Proof.* Set  $u = w$ ,  $v = w^{-\frac{1}{q-1}}$ ,  $\beta = \alpha(q-1)$ . By hypothesis  $0 < \alpha < 1/2$  and also  $0 < \alpha < 1/2(q-1)$  which implies that  $0 < \beta < 1/2$ , we can now use Lemma 3.4 to show that  $\mu_I^{q,\alpha}$  is a Carleson sequence with intensity at most  $c_\alpha[w]_{A_q^d}^\alpha$ . For the second statement suffices to notice that  $\nu_I^q \leq \mu_I^{q,\alpha}[w]_{A_q^d}^{1-\alpha}$  for all  $I \in \mathcal{D}$ , for some  $\alpha \in (0, \max\{1/2, 1/2(q-1)\})$   $\square$

A proof of this lemma for  $q = 2$  that works on geometric doubling metric spaces can be found in [NV1, V]. In those papers  $\alpha = 1/4$  can be used, and in that case the constant  $C_\alpha$  can be replaced by 288.

**3.2. Lift Lemma.** Given a dyadic interval  $L$ , and weights  $u, v$ , we introduce a family of stopping time intervals  $\mathcal{ST}_L^m$  such that the averages of the weights over any stopping time interval  $K \in \mathcal{ST}_L^m$  are comparable to the averages on  $L$ , and  $|K| \geq 2^m |L|$ . This construction appeared in [NV] for the case  $u = w$ ,  $v = w^{-1}$ . We also present a lemma that lifts  $w$ -Carleson sequences on intervals to  $w$ -Carleson sequences on “ $m$ -stopping intervals”. This was used in [NV] for a very specific choice of  $m$ -stopping time intervals  $\mathcal{ST}_L^m$ .

**Lemma 3.6** (Lift Lemma [NV]). *Let  $u$  and  $v$  be weights,  $L$  be a dyadic interval and  $m, n$  be fixed positive integers. Let  $\mathcal{ST}_L^m$  be the collection of maximal stopping time intervals  $K \in \mathcal{D}(L)$ , where the stopping criteria are either (i)  $|\Delta_K u|/m_K u + |\Delta_K v|/m_K v \geq 1/m + n + 2$ , or (ii)  $|K| = 2^{-m}|L|$ . Then for any stopping interval  $K \in \mathcal{ST}_L^m$ ,  $e^{-1}m_L u \leq m_K u \leq e m_L u$ , and hence also  $e^{-1}m_L v \leq m_K v \leq e m_L v$ .*

Note that the roles of  $m$  and  $n$  can be interchanged and we get the family  $\mathcal{ST}_L^n$  using the same stopping condition (i) and condition (ii) replaced by  $|K| = 2^{-n}|L|$ . Notice that  $\mathcal{ST}_L^m$  is a partition of  $L$  in dyadic subintervals of length at least  $2^{-m}|L|$ . The following lemma lifts a  $w$ -Carleson sequence to  $m$ -stopping time intervals with comparable intensity. For the particular  $m$ -stopping time  $\mathcal{ST}_L^m$  given by the stopping criteria (i) and (ii) in Lemma 3.6, and  $w = 1$ , this appeared in [NV].



**Lemma 3.7.** *For each  $L \in \mathcal{D}$  let  $\mathcal{ST}_L^m$  be a partition of  $L$  in dyadic subintervals of length at least  $2^{-m}|L|$ . Assume  $\{\nu_I\}_{I \in \mathcal{D}}$  is a  $w$ -Carleson sequence with intensity at most  $A$ , let  $\nu_L^m := \sum_{K \in \mathcal{ST}_L^m} \nu_K$ , then  $\{\nu_L^m\}_{L \in \mathcal{D}}$  is a  $w$ -Carleson sequence with intensity at most  $(m+1)A$ .*

For proofs you can see [MoP].

**3.3. Auxiliary quantities.** For a weight  $v$ , and a locally integrable function  $\phi$  we define the following quantities,

$$(3.3) \quad P_L^m \phi := \sum_{I \in \mathcal{D}_m(L)} |\langle \phi, h_I \rangle| \sqrt{|I|/|L|},$$

$$(3.4) \quad S_L^{v,m} \phi := \sum_{J \in \mathcal{D}_m(L)} |\langle \phi, h_J^v \rangle_v| \sqrt{m_J v} \sqrt{|J|/|L|},$$

$$(3.5) \quad R_L^{v,m} \phi := \sum_{J \in \mathcal{D}_m(L)} \frac{|\Delta_{J^v}|}{m_J v} m_J (|\phi|v) |J|/\sqrt{|L|},$$

Let  $w \in A_q^d$ ,  $\mathcal{ST}_L^m$  be an  $m$ -stopping time family of subintervals of  $L$ ,  $0 < \alpha < \max\{1/2, 1/2(q-1)\}$ , and  $\{\mu_K^q = \mu_K^{q,\alpha}\}_{K \in \mathcal{D}}$  be the Carleson sequence with intensity  $C_\alpha[w]_{A_q^d}$  defined in Lemma 3.5. For each  $m > 0$ , we introduce another sequence  $\{\mu_L^m\}$ , which is Carleson by Lemma 3.7:

$$\mu_L^m := \sum_{K \in \mathcal{ST}_L^m} \mu_K^q \quad \text{with intensity} \quad C_\alpha(m+1)[w]_{A_q^d}.$$

We will use the following estimates for  $S_L^{v,m} \phi$  and  $R_L^{v,m} \phi$ , where  $1 < p < 2$  will be dictated by the proof of the theorem.

$$(3.6) \quad S_L^{v,m} \phi \leq \left( \sum_{J \in \mathcal{D}_m(L)} |\langle \phi, h_J^v \rangle_v|^2 \right)^{\frac{1}{2}} (m_L v)^{\frac{1}{2}},$$

$$(3.7) \quad R_L^{v,m} \phi \leq C C_m^n (m_L v^{\frac{-1}{q-1}})^{\frac{-(q-1)}{2}} (m_L v)^{\frac{1}{2}} \inf_{x \in L} (M_{w^{-1}}(|g|^p)(x))^{\frac{1}{p}} \sqrt{\mu_L^m},$$

See [NV] for the proof when  $q = 2$ , slight modification of their argument gives the estimate for  $R_L^{v,m} \phi$ . Estimating  $P_L^m \phi$  is very simple:

$$(3.8) \quad (P_L^m \phi)^2 \leq \sum_{I \in \mathcal{D}_m(L)} |I|/|L| \sum_{I \in \mathcal{D}_m(L)} |\langle \phi, h_I \rangle|^2 = \sum_{I \in \mathcal{D}_m(L)} |\langle \phi, h_I \rangle|^2.$$

*Remark 3.8.* In [NV1], Nazarov and Volberg extend the results that they had in [NV] for Haar shifts to metric spaces with geometric doubling. Following the same modifications in the argument made from [NV] to [NV1], one could obtain the same result as in Theorem 4.1 on a metric space with geometric doubling, see [Mo1].

## 4. HAAR MULTIPLIERS

For a weight  $w$ ,  $t \in \mathbb{R}$ , and  $m, n \in \mathbb{N}$ , a  $t$ -Haar multiplier of complexity  $(m, n)$  is the operator defined as

$$(4.1) \quad T_{t,w}^{m,n} f(x) := \sum_{L \in \mathcal{D}} \sum_{I \in \mathcal{D}_n(L); J \in \mathcal{D}_m(L)} \frac{\sqrt{|I||J|}}{|L|} \left( \frac{w(x)}{m_L w} \right)^t \langle f, h_I \rangle h_J(x).$$

In [MoP] it is shown that  $w \in C_{2t}^d$  is a necessary condition for boundedness of  $T_{w,t}^{m,n}$  in  $L^2(\mathbb{R})$ . It is also shown that the  $C_{2t}^d$ -condition is sufficient for a  $t$ -Haar multiplier of complexity  $(m, n)$  to be bounded in  $L^2(\mathbb{R})$  for most  $t$ ; this was proved in [KP] for the case  $m = n = 0$ . Here we are concerned not only with the boundedness but also with the dependence of the operator norm on the  $C_{2t}^d$ -constant. For  $T_w^t$  and  $t = 1, \pm 1/2$  this was studied in [P2]. The first author [Be] was able to obtain estimates, under the additional condition on the weight  $w^{2t} \in A_q^d$  for some  $q > 1$ , for  $T_w^t$  and for all  $t \in \mathbb{R}$ . Her results were generalized for  $T_{w,t}^{m,n}$  for all  $t$  when  $w^{2t} \in A_2^d$ , see [MoP]. We will show that:

**Theorem 4.1.** *Let  $t$  be a real number and  $w$  a weight such that  $w^{2t} \in A_q^d$  for some  $q > 1$  (i.e.  $w^{2t} \in A_\infty^d$ ), then*

$$\|T_{t,w}^{m,n} f\|_2 \leq C_q (m + n + 2)^3 [w]_{C_{2t}^d}^{\frac{1}{2}} [w^{2t}]_{A_q^d}^{\frac{1}{2}} \|f\|_2.$$

Using Lemmas 2.1 and 2.3 we can refine the result as follows, where  $C_m^n = n + m + 2$ .

**Theorem 4.2.** *Let  $t \in \mathbb{R}$ ,  $w \in C^{2t}$  then*

(i) *If  $0 < 2t < 1$  and  $w \in A_p^d$  then*

$$\|T_{t,w}^{m,n} f\|_2 \leq C_p (C_m^n)^3 [w^{2t}]_{A_p^d}^{\frac{1}{2}} \|f\|_2 \leq C_p (C_m^n)^3 [w]_{A_p^d}^t \|f\|_2.$$

(ii) *If  $t > 1$  and  $w \in A_p^d$  then if  $q = 2t(p - 1) + 1$*

$$\|T_{t,w}^{m,n} f\|_2 \leq C_p (C_m^n)^3 [w]_{C_{2t}^d}^{\frac{1}{2}} [w^{2t}]_{A_q^d}^{\frac{1}{2}} \|f\|_2 \leq C_p (C_m^n)^3 [w]_{C_{2t}^d} [w]_{A_p^d}^t.$$

(iii) *If  $t < 0$  then*

$$\|T_{t,w}^{m,n} f\|_2 \leq C (C_m^n)^3 [w]_{C_{2t}^d} \|f\|_2 = C (C_m^n)^3 [w]_{A_{1-1/2t}^d}^{-2t} \|f\|_2.$$

*Remark 4.3.* Throughout the proof a constant  $C_q$  will be a numerical constant depending only on the parameter  $q > 1$  that may change from line to line.

*Proof of Theorem 4.2.* By Lemma 2.3 if  $w \in C_{2t}^d \cap A_\infty^d$  then there is  $q > 1$  such that  $w^{2t} \in A_q^d$ , matching cases perfectly. Now use Theorem 4.1.  $\square$

*Proof of Theorem 4.1.* Fix  $f, g \in L^2(\mathbb{R})$ . By duality, it is enough to show that

$$|\langle T_{t,w}^{m,n} f, g \rangle| \leq C(m+n+2)^3 [w]_{C_{2t}^d}^{\frac{1}{2}} [w^{2t}]_{A_q^d}^{\frac{1}{2}} \|f\|_2 \|g\|_2.$$

The inner product on the left-hand-side can be expanded into a double sum, that we now estimate,

$$|\langle T_{t,w}^{m,n} f, g \rangle| \leq \sum_{L \in \mathcal{D}} \sum_{I \in \mathcal{D}_n(L); J \in \mathcal{D}_m(L)} \frac{\sqrt{|I||J|} |\langle f, h_I \rangle|}{|L| (m_L w)^t} |\langle g w^t, h_J \rangle|.$$

Write  $h_J$  as a linear combination of a weighted Haar function and a characteristic function,  $h_J = \alpha_J h_J^{w^{2t}} + \beta_J \chi_J / \sqrt{|J|}$ , where  $\alpha_J = \alpha_J^{w^{2t}}$ ,  $\beta_J = \beta_J^{w^{2t}}$ ,  $|\alpha_J| \leq \sqrt{m_J w^{2t}}$ , and  $|\beta_J| \leq |\Delta_J(w^{2t})| / m_J w^{2t}$ . Now break into two terms to be estimated separately so that,

$$|\langle T_{t,w}^{m,n} f, g \rangle| \leq \Sigma_1^{m,n} + \Sigma_2^{m,n},$$

where

$$\begin{aligned} \Sigma_1^{m,n} &:= \sum_{L \in \mathcal{D}} \sum_{I \in \mathcal{D}_n(L); J \in \mathcal{D}_m(L)} \frac{\sqrt{|I||J|} \sqrt{m_J(w^{2t})}}{|L| (m_L w)^t} |\langle f, h_I \rangle| |\langle g w^t, h_J^{w^{2t}} \rangle|, \\ \Sigma_2^{m,n} &:= \sum_{L \in \mathcal{D}} \sum_{I \in \mathcal{D}_n(L); J \in \mathcal{D}_m(L)} \frac{|J| \sqrt{|I|} |\Delta_J(w^{2t})|}{|L| (m_L w)^t m_J(w^{2t})} |\langle f, h_I \rangle| m_J(|g| w^t). \end{aligned}$$

Let  $p = 2 - (C_n^m)^{-1}$  (note that  $2 > p > 1$ , in fact is getting closer to 2 as  $m$  and  $n$  increase), and define as in (3.3), (3.4) and (3.5), the quantities  $P_L^m \phi$ ,  $S_L^{v,n} \phi$  and  $R_L^{v,n} \phi$ , we will use here the case  $v = w^{2t}$ , for appropriate  $\phi$ s and corresponding estimates. Note that  $1 < p < 2$ .

The sequence  $\{\eta_I\}_{I \in \mathcal{D}}$  where

$$\eta_I := (m_I w^{2t}) (m_I w^{\frac{-2t}{q-1}})^{(q-1)} \left( \frac{|\Delta_I(w^{2t})|^2}{|m_I w^{2t}|^2} + \frac{|\Delta_I(w^{-2t/(q-1)})|^2}{|m_I w^{-2t/(q-1)}|^2} \right) |I|,$$

is a Carleson sequence with intensity  $C_q [w^{2t}]_{A_q^d}$  by Lemma 3.5. The sequence  $\{\eta_L^m\}_{L \in \mathcal{D}}$  where

$$\eta_L^m := \sum_{I \in \mathcal{ST}_L^m} \eta_I,$$

and the stopping time  $\mathcal{ST}_L^m$  is defined as in Lemma 3.6 but with respect to the weights  $u = w^{2t}$ ,  $v = w^{-2t/(q-1)}$ , is a Carleson sequence with intensity  $C_q(m+1)[w^{2t}]_{A_q^d}$  by Lemma 3.7, .

Observe that on the one hand  $\langle gw^t, h_J^{w^{2t}} \rangle = \langle gw^{-t}, h_J^{w^{2t}} \rangle_{w^{2t}}$ , and on the other  $m_J(|g|w^t) = m_J(|g|w^{-t}|w^{2t})$ . Therefore,

$$\Sigma_3^{m,n} = \sum_{L \in \mathcal{D}} (m_L w)^{-t} S_L^{w^{2t},n}(gw^{-t}) P_L^m f,$$

$$\Sigma_4^{m,n} = \sum_{L \in \mathcal{D}} (m_L w)^{-t} R_L^{w^{2t},n}(gw^{-t}) P_L^m f.$$

Estimates (3.6) and (3.7) hold for  $S_L^{w^{2t},m}(gw^{-t})$  and  $R_L^{w^{2t},m}(gw^{-t})$  with  $v$  and  $\phi$  replaced by  $w^{2t}$  and  $gw^{-t}$ :

$$S_L^{w^{2t},n}(gw^{-t}) \leq (m_L w^{2t})^{\frac{1}{2}} \left( \sum_{J \in \mathcal{D}_m(L)} |\langle gw^{-t}, h_J^{w^{2t}} \rangle_{w^{2t}}|^2 \right)^{\frac{1}{2}},$$

$$R_L^{w^{2t},n}(gw^{-t}) \leq C C_m^n (m_L w^{2t})^{\frac{1}{2}} (m_L w^{\frac{-2t}{q-1}})^{\frac{-(q-1)}{2}} F^{\frac{1}{2}}(x) \sqrt{\eta_L^m},$$

where  $F(x) = \inf_{x \in L} (M_{w^{2t}}(|gw^{-t}|^p)(x))^{\frac{2}{p}}$ .

**Estimating  $\Sigma_1^{m,n}$ :** Plug in the estimates for  $S_L^{w^{2t},n}(gw^{-t})$  and  $P_L^m f$ , observe that  $(m_L w^{2t})^{\frac{1}{2}} / (m_L w)^t \leq [w]_{C_{2t}^d}^{\frac{1}{2}}$ , use the Cauchy-Schwarz inequality, to get,

$$\begin{aligned} \Sigma_1^{m,n} &\leq \sum_{L \in \mathcal{D}} [w]_{C_{2t}^d}^{\frac{1}{2}} \left( \sum_{J \in \mathcal{D}_n(L)} |\langle gw^{-t}, h_J^{w^{2t}} \rangle_{w^{2t}}|^2 \right)^{\frac{1}{2}} \left( \sum_{I \in \mathcal{D}_m(L)} |\langle f, h_I \rangle|^2 \right)^{\frac{1}{2}} \\ &\leq [w]_{C_{2t}^d}^{\frac{1}{2}} \|f\|_2 \left( \sum_{L \in \mathcal{D}} \sum_{J \in \mathcal{D}_n(L)} |\langle gw^{-t}, h_J^{w^{2t}} \rangle_{w^{2t}}|^2 \right)^{\frac{1}{2}} \\ &\leq [w]_{C_{2t}^d}^{\frac{1}{2}} \|f\|_2 \|gw^{-t}\|_{L^2(w^{2t})} = [w]_{C_{2t}^d}^{\frac{1}{2}} \|f\|_2 \|g\|_2. \end{aligned}$$

**Estimating  $\Sigma_2^{m,n}$ :** Plug in the estimates for  $R_L^{w^{2t},n}(gw^{-t})$  and  $P_L^m f$ , where  $F(x) = (M_{w^{2t}}(|gw^{-t}|^p)(x))^{2/p}$ , use the Cauchy-Schwarz inequality and  $(m_L w^{2t})^{\frac{1}{2}} / (m_L w)^t \leq [w]_{C_{2t}^d}^{\frac{1}{2}}$  to get

$$\Sigma_2^{m,n} \leq C C_m^n [w]_{C_{2t}^d}^{\frac{1}{2}} \|f\|_2 \left( \sum_{L \in \mathcal{D}} (\eta_L^m / (m_L w^{\frac{-2t}{q-1}})^{q-1}) \inf_{x \in L} F(x) \right)^{\frac{1}{2}}.$$

Now using Weighted Carleson Lemma 3.1 with  $\alpha_L = \eta_L^m / (m_L w^{\frac{-2t}{q-1}})^{q-1}$  (which by Lemma 3.2 is a  $w^{2t}$ -Carleson sequence with intensity no larger than  $C_q(m+1)[w]_{A_q^d}$ ,  $F(x) = (M_{w^{2t}}(|gw^{-t}|^p)(x))^{2/p}$ , and  $v = w^{2t}$ ,

$$\Sigma_2^{m,n} \leq C_q (C_m^n)^2 [w]_{C_{2t}^d}^{\frac{1}{2}} [w^{2t}]_{A_q^d}^{\frac{1}{2}} \|f\|_2 \left\| M_{w^{2t}}(|gw^{-t}|^p) \right\|_{L^{\frac{2}{p}}(w^{2t})}^{\frac{1}{p}}.$$

Using (2.1), that is the boundedness of  $M_{w^{2t}}$  in  $L^{\frac{2}{p}}(w^{2t})$  for  $2/p > 1$ ,

$$\begin{aligned} \Sigma_2^{m,n} &\leq C_q (C_m^n)^2 (2/p)' [w]_{C_{2t}^d}^{\frac{1}{2}} [w^{2t}]_{A_q^d}^{\frac{1}{2}} \|f\|_2 \left\| |gw^{-t}|^p \right\|_{L^{\frac{2}{p}}(w^{2t})}^{\frac{1}{p}} \\ &\leq C_q (C_m^n)^3 [w]_{C_{2t}^d}^{\frac{1}{2}} [w^{2t}]_{A_q^d}^{\frac{1}{2}} \|f\|_2 \|g\|_2, \end{aligned}$$

Since  $(2/p)' = 2/(2-p) = 2C_m^n$ . The theorem is proved.  $\square$

### APPENDIX

*Proof of Lemma 3.2.* We will show this inequality using a Bellman function type method. Consider  $B(u, v, l) := u - 1/(v^{p-1}(1+l))$  defined on the domain  $\mathbb{D} = \{(u, v, l) \in \mathbb{R}^3, u > 0, v > 0, uv^{p-1} > 1 \text{ and } 0 \leq l \leq 1\}$ . Note that  $\mathbb{D}$  is convex. Note that

$$(4.2) \quad 0 \leq B(u, v, l) \leq u \quad \text{for all } (u, v, l) \in \mathbb{D}$$

and

$$(4.3) \quad (\partial B / \partial l)(u, v, l) \geq 1/4v^{p-1} \quad \text{for all } (u, v, l) \in \mathbb{D}.$$

and also  $-(du, dv, dl)d^2B(du, dv, dl)^t$  is non-negative because, it equals

$$\begin{aligned} &-(du, dv, dl) \begin{pmatrix} 0 & 0 & 0 \\ 0 & p(1-p)\frac{v^{-p-1}}{1+l} & (1-p)\frac{v^{-p}}{(l+1)^2} \\ 0 & (1-p)\frac{v^{-p}}{(l+1)^2} & -2\frac{v^{1-p}}{(l+1)^3} \end{pmatrix} \begin{pmatrix} du \\ dv \\ dl \end{pmatrix} \\ &= p(p-1)\frac{v^{-p-1}}{1+l}(du)^2 + 2(p-1)\frac{v^{-p}}{(l+1)^2}dudv + 2\frac{v^{1-p}}{(l+1)^3}(dv)^2 \geq 0, \end{aligned}$$

since all terms are positive for  $p > 1$ .

Now let us show that if  $(u_-, v_-, l_-)$  and  $(u_+, v_+, l_+)$  are in  $\mathbb{D}$  and we define  $(u_0, v_0, l) \in \mathbb{D}$  where  $l$  is in between  $l_+$  and  $l_-$ ,  $u_0 = (u_- + u_+)/2$ ,  $v_0 = (v_- + v_+)/2$ , and  $l_0 = (l_- + l_+)/2$ , then

$$B(u_0, v_0, l) - (B(u_-, v_-, l_-) + B(u_+, v_+, l_+)/2) \geq |l - l_0|/4v_0^{p-1}$$

Write for  $-1 \leq t \leq 1$ ,  $u(t) = [(t+1)u_+ + (1-t)u_-]/2$ ,  $v(t) = [(t+1)v_+ + (1-t)v_-]/2$ , and  $l(t) = [(t+1)l_+ + (1-t)l_-]/2$ . Define  $b(t) := B(u(t), v(t), l(t))$ , then  $b(0) = B(u_0, v_0, l_0)$ ,  $b(1) = B(u_+, v_+, l_+)$ ,  $b(-1) = B(u_-, v_-, l_-)$ ,  $du/dt = (u_+ - u_-)/2$ ,  $dv/dt = (v_+ - v_-)/2$  and  $dl/dt = (l_+ - l_-)/2$ . If  $(u_+, v_+, l_+)$  and  $(u_-, v_-, l_-)$  are in  $\mathbb{D}$  then  $(u(t), v(t), l(t))$  is also in  $\mathbb{D}$  for all  $|t| \leq 1$ , since  $\mathbb{D}$  is convex. It is a calculus exercise to show that

$$(4.4) \quad b(0) - \frac{b(1) + b(-1)}{2} = \frac{-1}{2} \int_{-1}^1 (1 - |t|)b''(t)dt$$

Also it is easy to check that  $-b''(t) = -\left(\frac{du}{dt}, \frac{dv}{dt}, \frac{dl}{dt}\right) d^2 B\left(\frac{du}{dt}, \frac{dv}{dt}, \frac{dl}{dt}\right)^t$ . By the Mean Value Theorem and (4.4),

$$\begin{aligned} B(u_0, v_0, l) &= \frac{B(u_-, v_-, l_-) + B(u_+, v_+, l_+)}{2} \\ &= (l - l_0) \frac{\partial B}{\partial l}(u_0, v_0, l') - \frac{1}{2} \int_{-1}^1 (1 - |t|) b''(t) dt \geq \frac{l - l_0}{4v_0^{p-1}}, \end{aligned}$$

where  $l'$  is a point between  $l$  and  $l_0 = (l_- + l_+)/2$ .

Now we can use the Bellman function argument. Let  $u_+ = m_{J_+} w$ ,  $u_- = m_{J_-} w$ ,  $v_+ = m_{J_+} w^{\frac{-1}{p-1}}$ ,  $v_- = m_{J_-} v^{\frac{-1}{p-1}}$ ,  $l_+ = \frac{1}{|J_+|Q} \sum_{I \in \mathcal{D}(J_+)} \lambda_I$  and  $l_- = \frac{1}{|J_-|Q} \sum_{I \in \mathcal{D}(J_-)} \lambda_I$ . Thus  $(u_-, v_-, l_-), (u_+, v_+, l_+) \in \mathbb{D}$  and  $u_0 = m_J w$ ,  $v_0 = m_J w^{\frac{-1}{p-1}}$ , and  $l_0 = \frac{1}{|J|Q} \sum_{I \in \mathcal{D}(J)} \lambda_I$ . Thus

$$(u_0, v_0, l_0) - ((u_- + u_+)/2, (v_- + v_+)/2, (l_- + l_+)/2) = (0, 0, \lambda_J/Q|J|).$$

Then we can run the usual induction on scale arguments using the properties of the Bellman function,

$$\begin{aligned} |J| m_J w &\geq |J| B(u_0, v_0, l_0) \\ &\geq |J| \frac{B(u_+, v_+, l_+)}{2} + |J| \frac{B(u_-, v_-, l_-)}{2} + \lambda_J/4Q (m_J w^{\frac{-1}{p-1}})^{p-1} \\ &= |J_+| B(u_+, v_+, l_+) + |J_-| B(u_-, v_-, l_-) + \lambda_J/4Q (m_J w^{\frac{-1}{p-1}})^{p-1} \end{aligned}$$

Iterating, we get

$$m_J w \geq \frac{1}{4Q|J|} \sum_{I \in \mathcal{D}(J)} \frac{\lambda_I}{(m_I w^{-1/p-1})^{p-1}}.$$

□

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